MEE5114 Advanced Control for Robotics

Lecture 11: Elementary Model Predictive Control

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Outline

- Formulation of General MPC Problems
- Linear MPC Problems
- Linear MPC Example: Cessna Citation Aircraft

MPC Formulation: System Model

- General discrete-time nonlinear systems:

system
robot

$$\begin{cases} x(t+1) = f(x(t), u(t)) \\ y(t) = h(x(t)) \end{cases} \qquad t \in \mathbb{Z}_+ \tag{1}$$

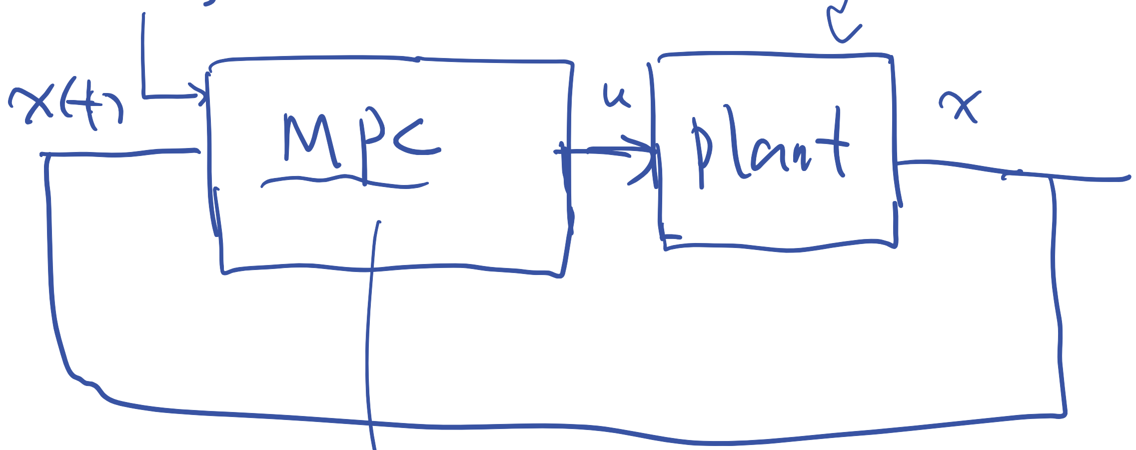
- State and Control constraints:

model of plant

$$x(t) \in \mathcal{X} \quad u(t) \in \mathcal{U} \tag{2}$$

- For simplicity, we assume full state information is available (e.g. $y(t) = x(t)$ or $h(\cdot)$ is bijection).

reference



$$u(t) = \mu_{mpc}(x(t))$$

MPC Formulation: Online Optimization Problem

→ optimization

- At time t: solve the following N-horizon optimal control problem:

terminal cost

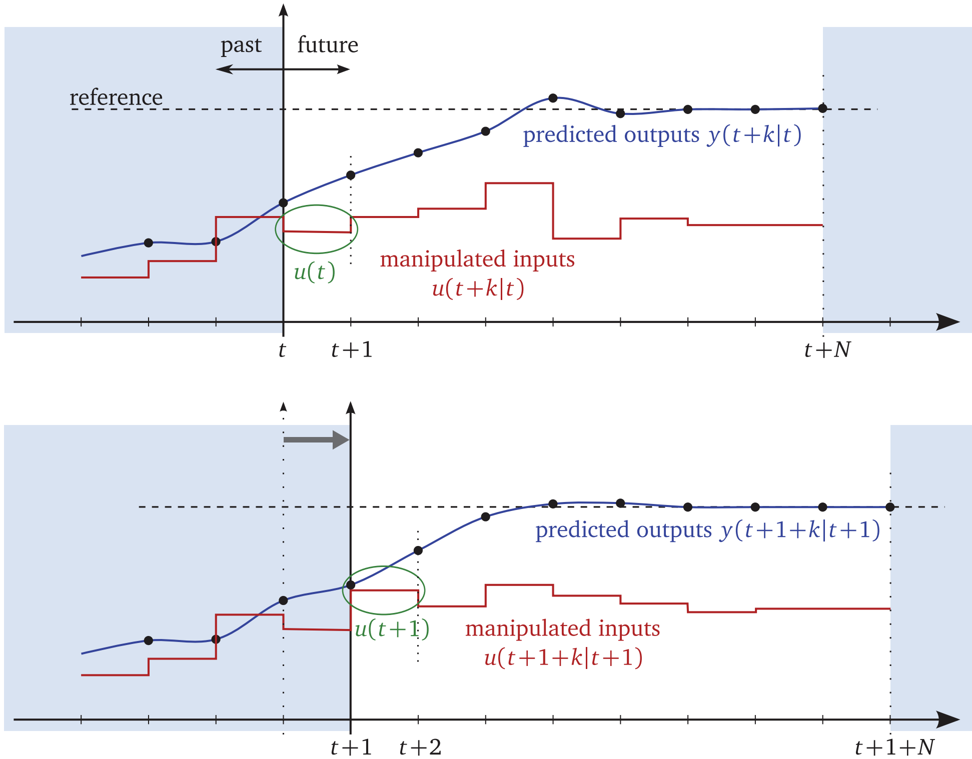
$$\mathcal{P}_N(x(t)): \quad V_N(x(t)) = \begin{cases} \min_{U_0} & J_N(x(t), U_0) \triangleq J_f(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k) \\ \text{subj. to:} & x_{k+1} = f(x_k, u_k), k = 0, \dots, N-1 \\ & x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f, \quad x_0 = x(t) \end{cases} \tag{3}$$

running cost

optimization problem
that depends on current state

- $U_0 = (u_0, u_1, \dots, u_{N-1})$: overall control vector
- $\mathcal{X}_f \subseteq \mathcal{X}$: terminal state constraint set
- Assume nonnegative cost functions: $J_f : \mathcal{X} \to \mathbb{R}_+$ and $l : \mathcal{X} \times \mathcal{U} \to \mathbb{R}_+$
- Given $x(t)$ at time t, optimal control sequence $u_0^*, \dots, u_{N-1}^*$ can be found via numerical optimization

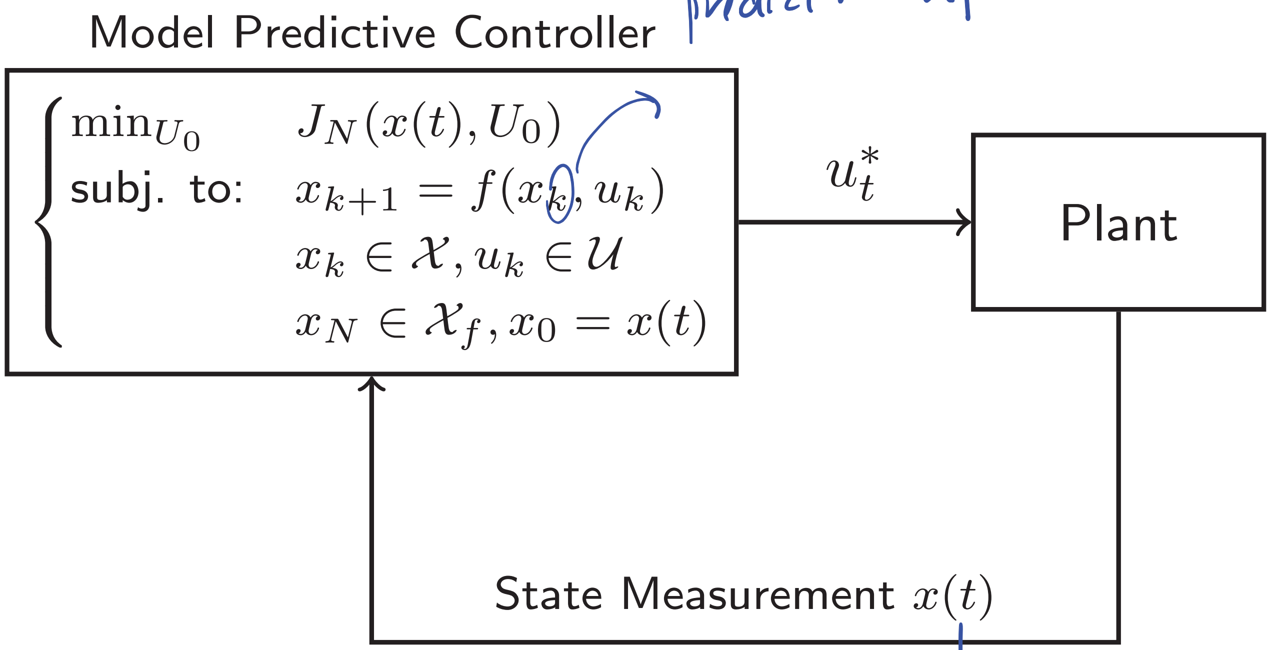
MPC Formulation: Receding Horizon Implementation



- At time t, solves an N-horizon optimal control problem (3)
- Apply the first step of the optimal control sequence
- At time $t+1$, horizon is shifted and the optimal control problem is solved again using newly obtained state information

MPC Diagram

prediction step



t: current time

Main Topics of MPC

- Computational Issues:
  - Standard MPC forms and their online optimization algorithms (this lecture)
  - Explicit MPC: solve the optimal MPC control law offline to simply online computation
- Theoretical Issues
  - Recursive feasibility
  - Closed-loop stability : ← DP (Dynamic Programming theory + Lyapunov)

    Value function
- Other topics
  - Distributed MPC
  - Stochastic MPC
  - Embedded MPC
  - ⋮

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- Linear MPC Example: Cessna Citation Aircraft

MPC of Linear Systems

- Linear MPC problem: linear dynamical system + polyhedral state/control constraints + convex cost functions

optimization

- At time t: solve the following N-horizon optimal control problem:

$$\mathcal{P}_N(x(t)): \quad V_N(x(t)) = \begin{cases} \min_{U_0} & J_N(x(t), U_0) \triangleq J_f(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k) \\ \text{subj. to:} & x_{k+1} = Ax_k + Bu_k, k = 0, \ldots, N-1 \\ & A_x x_k \leq b_x, A_u u_k \leq b_u, k = 0, \ldots, N-1 \\ & A_f x_N \leq b_f, \quad x_0 = x(t) \end{cases} \tag{4}$$

optimization variable: $U_0 = \{u_0, u_1 \cdots u_{N-1}\} \in \mathbb{R}^{N \cdot m \times 1}$

- Cost functions $J_N(x(t), U_0)$: 2-norm, 1-norm, ∞-norm:
  - 2-norm: $x_N^T Q_f x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)$ ✓ Let's work with this cost
  - $1/\infty$-norm: $\|Q_f x_N\|_p + \sum_{k=0}^{N-1} (\|Q x_k\|_p + \|R u_k\|_p)$, $p = 1, \infty$, Q, Q_f, R are full column rank matrices
  - Recall: $\|x\|_1 = \sum_i |x_i|$ and $\|x\|_\infty = \max_i |x_i|$

- cost & constraints depend on $x_k, u_k, \; k=0,\dots,N-1, \; x_N$

- Decision variable $U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$

x_k, depends on $\begin{cases} x_0 = x(t) \Leftarrow \text{given} \\ \underline{u_0, \dots, u_{k-1}} \end{cases}$

- We want to write everything in terms of optimization variable.
   - eg optimization solver: $\min_u f(u)$
     $g(u) \le 0$

- $x_1 = Ax_0 + Bu_0$

$x_2 = Ax_1 + Bu_1 = A^2 x_0 + ABu_0 + Bu_1$

$\vdots$

$A \in \mathbb{R}^{n \times n}$

$U_0 \in \mathbb{R}^{N \cdot m \times 1}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\mathbb{R}^{N \cdot n \times n}} \underbrace{x(t)}_{\text{given},\ \mathbb{R}^n \to \text{state - dim}} + \underbrace{\begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & \\ A^2B & AB & B & & \\ \vdots & & & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & \cdots & B \end{bmatrix}}_{\mathbb{R}^{N \cdot n \times N \cdot m}} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$\|$ $\qquad$ $\|$

$$\underbrace{\qquad}_{\displaystyle X} \qquad \underbrace{\qquad}_{\displaystyle S_x} \qquad \underbrace{\qquad\qquad}_{\displaystyle S_u} \qquad \underbrace{\qquad}_{\displaystyle U_0}$$

$$X = S_x \cdot x(t) + S_u U_0$$

$$\left(\sum_{k=1}^{N-1} x_k^T Q x_k + x_N^T Q_f x_N \right) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}^T \underbrace{\begin{bmatrix} Q & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & Q_f \end{bmatrix}}_{} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$X^T \tilde{Q} X$$

$$\sum_{k=0}^{N-1} u_k^T R u_k = U_0^T \tilde{R} U_0, \qquad \tilde{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}$$

$$J_N(x(t), U_0) = X^T \tilde{Q} X + U_0^T \tilde{R} U_0$$

$$= (S_x \cdot x(t) + S_u \cdot U_0)^T \tilde{Q} (\quad) + U_0^T \tilde{R} U_0$$

$$= U_0^T \big(\underbrace{\tilde{R} + S_u^T \tilde{Q} S_u}_{H} \big) U_0 + 2\, x(t)^T \underbrace{S_x^T \tilde{Q} S_u}_{F} \cdot U_0 + (\text{constant})$$

Batch Formulation of Linear MPC (1/4)

- Vector form of prediction model over N horizon:

$$\underbrace{\begin{bmatrix} x(t) \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} I \\ A \\ \vdots \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{S}^x} x(t) + \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & \cdots & \cdots & B \end{bmatrix}}_{\mathcal{S}^u} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}}_{U_0} \tag{5}$$

- Define $X = \mathcal{S}^x x(t) + \mathcal{S}^u U_0$
- 2-norm cost function becomes: $J_N(x(t), U_0) = X^T \bar{Q} X + {U_0}^T \bar{R} U_0$,
  - $\bar{Q} \triangleq \text{diag}\{Q, \ldots, Q, Q_f\}, \bar{Q} \succeq 0$
  - $\bar{R} \triangleq \text{diag}\{R, \ldots, R\}, \bar{R} \succ 0$

Batch Formulation of Linear MPC (2/4)

- Substituting the expression of X:

$$\begin{aligned}
J_N(x(t), U_0) &= (\mathcal{S}^x x(t) + \mathcal{S}^u U_0)^T \bar{Q} (\mathcal{S}^x x(t) + \mathcal{S}^u U_0) + {U_0}^T \bar{R} U_0 \\
&= {U_0}^T \underbrace{((\mathcal{S}^u)^T \bar{Q} \mathcal{S}^u + \bar{R})}_{H} U_0 + 2x^T(t) \underbrace{(\mathcal{S}^x)^T \bar{Q} \mathcal{S}^u}_{F} U_0 + x^T(t) \underbrace{((\mathcal{S}^x)^T \bar{Q} \mathcal{S}^x)}_{Y} x(t) \\
&= {U_0}^T H U_0 + 2x^T(t) F U_0 + x^T(t) Y x(t) \\
&= \left[U_0^T, x(t)^T\right] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} \begin{bmatrix} U_0 \\ x(t) \end{bmatrix}
\end{aligned}$$

constraints: $A_x x_k \le b_x$, $k = 1, 2 \ldots, N$

$$\underbrace{\begin{bmatrix} A_x & & & \\ & A_x & & \\ & & \ddots & \\ & & & A_x \end{bmatrix}}_{\tilde{A}_x} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{\underline{X}} \le \underbrace{\begin{bmatrix} b_x \\ b_x \\ \vdots \\ b_x \end{bmatrix}}_{\tilde{b}_x}$$

$$\tilde{A}_x (S_x \cdot x(t) + S_u U_0) \le \tilde{b}_x$$

$$(\tilde{A}_x S_u) U_0 \le \tilde{b}_x - \tilde{A}_x S_x \cdot x(t)$$

$A_u \cdot u_k \le b_u$, $k = 0, \ldots N-1$

$$\underbrace{\begin{bmatrix} A_u & & & \\ & A_u & & \\ & & \ddots & \\ & & & A_u \end{bmatrix}}_{\tilde{A}_u} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{U_0} \le \underbrace{\begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \end{bmatrix}}_{\tilde{b}_u}$$

$\Rightarrow$ putting them together

$$\Rightarrow \begin{bmatrix} \tilde{A}_x S_u \\ \tilde{A}_u \end{bmatrix} U_0 \le \begin{bmatrix} \tilde{b}_x - \tilde{A}_x S_x \, x(t) \\ \tilde{b}_u \end{bmatrix}$$

Batch Formulation of Linear MPC (3/4)

- Polyhedral constraints can be reduced to:

$$\boxed{G_0 U_0 \le w_0 + E_0 x(t)}$$

where $G_0, w_0, E_0, x(t)$ are known matrices/vectors at time t.

QP solver: $\min_{U_0} \frac{1}{2} U_0^T \Gamma U_0 + g^T U_0 + c$

sub: $H U_0 \le h$

$u^- \le U_0 \le u^+$

MPC $\Longrightarrow$ $\min_{U_0} U_0^T H U_0 + 2x(t)^T F \cdot U_0 + \underbrace{x(t)^T \Gamma x(t)}$

subj: $G_0 U_0 \le w_0 + E_0 x(t)$

Batch Formulation of Linear MPC (4/4)

- $\mathcal{P}_N(x(t))$ boils down to a quadratic programming (QP) problem:

$$V_N(x(t)) = \begin{cases} \min_{U_0} & J_N(x(t), U_0) = [U_0^T, x(t)^T] \begin{bmatrix} H & F^T \\ F & Y \end{bmatrix} \begin{bmatrix} U_0 \\ x(t) \end{bmatrix} \\ \text{subj. to} & G_0 U_0 \le w_0 + E_0 x(t) \end{cases} \tag{6}$$

  - Whenever $H \succ 0$, the above QP with affine constraints is a special convex optimization problem, which can be solved very efficiently
  - Optimization variable is U_0; all parameters are known and constant;
  - $x(t)$ is the measured state at each time t, which is also known before solving the QP.

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Linear MPC Example: Cessna Citation Aircraft

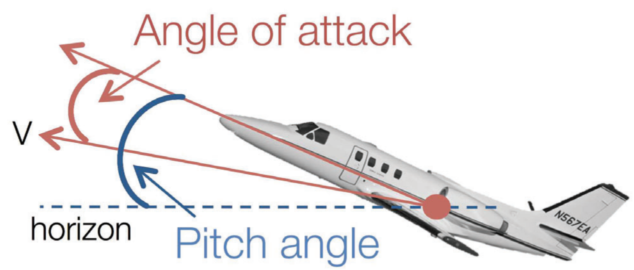
Example 1 (Cessna Citation Aircraft).

Linearized continuous-time model: (at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad $(\pm 15^\circ)$, elevator rate ± 0.349 rad/s $(\pm 20^\circ/s)$, pitch angle ± 0.650 rad $(\pm 37^\circ)$



- Open-loop response is unstable (open-loop poles: $0,\ 0,\ -1.5594 \pm 2.29i$)

Linear MPC Example: LQR vs. MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$V^*(x(t)) = \min_u \sum_{k=0}^{\infty} (x_t^T Q x_t + u_k^T R u_k)$$

s.t. $\quad x_{k+1} = Ax_k + Bu_k$

$\quad x_0 = x(t)$

No constraint:

optimal control law: $u^*(t) = -K_{LQR}^* \cdot x(t)$

MPC

$$V_N^*(x(t)) = \min_u \sum_{k=0}^{N-1} (x_t^T Q x_t + u_k^T R u_k)$$

s.t. $\quad x_{k+1} = Ax_k + Bu_k$

$\quad x_k \in \mathcal{X}, u_k \in \mathcal{U}$

$\quad x_0 = x(t)$

- Assume: $Q = Q^T \succeq 0,\ R = R^T \succ 0$

Riccati Equation

$\Rightarrow P^* \Rightarrow K_{LQR}^*$

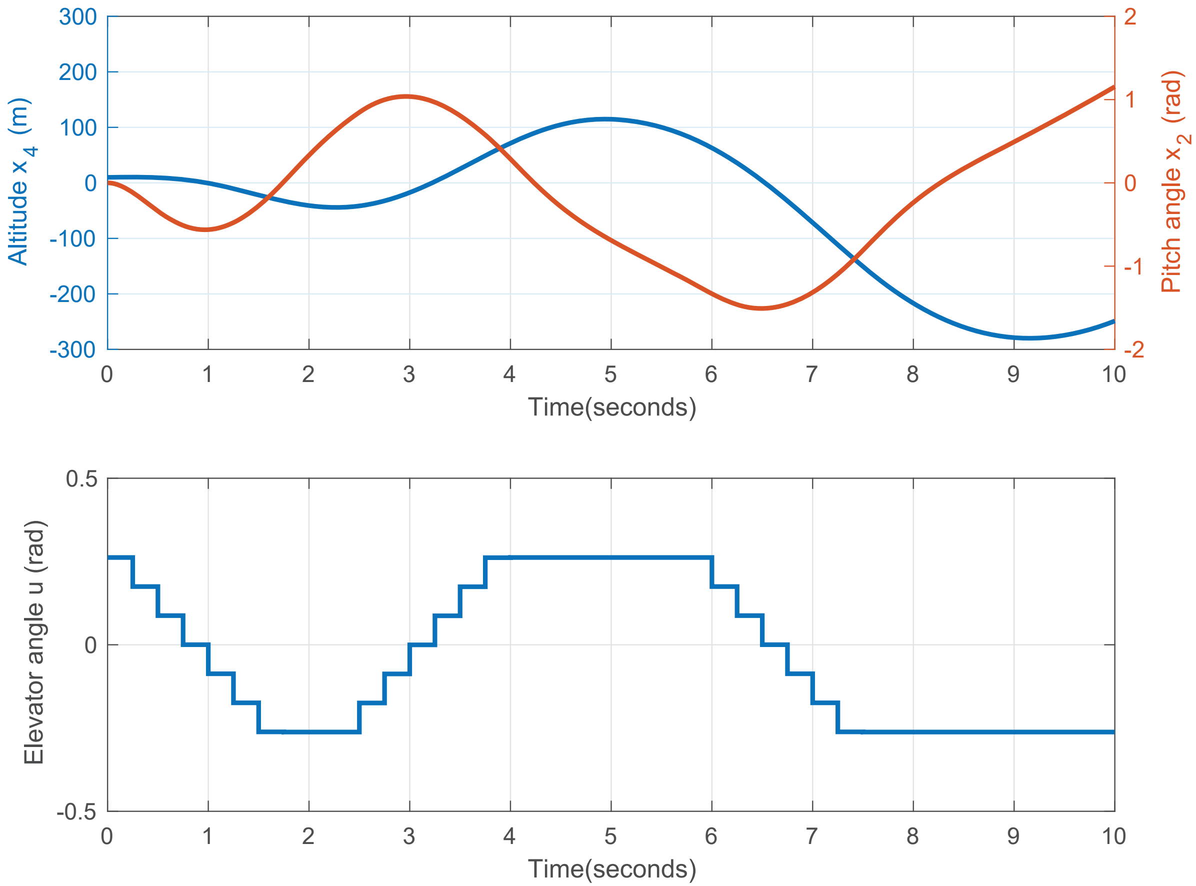
Linear MPC Example: LQR with Saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Let K_{LQR} be the optimal LQR gain

$(A - BK_{LQR})$ is Hurwitz



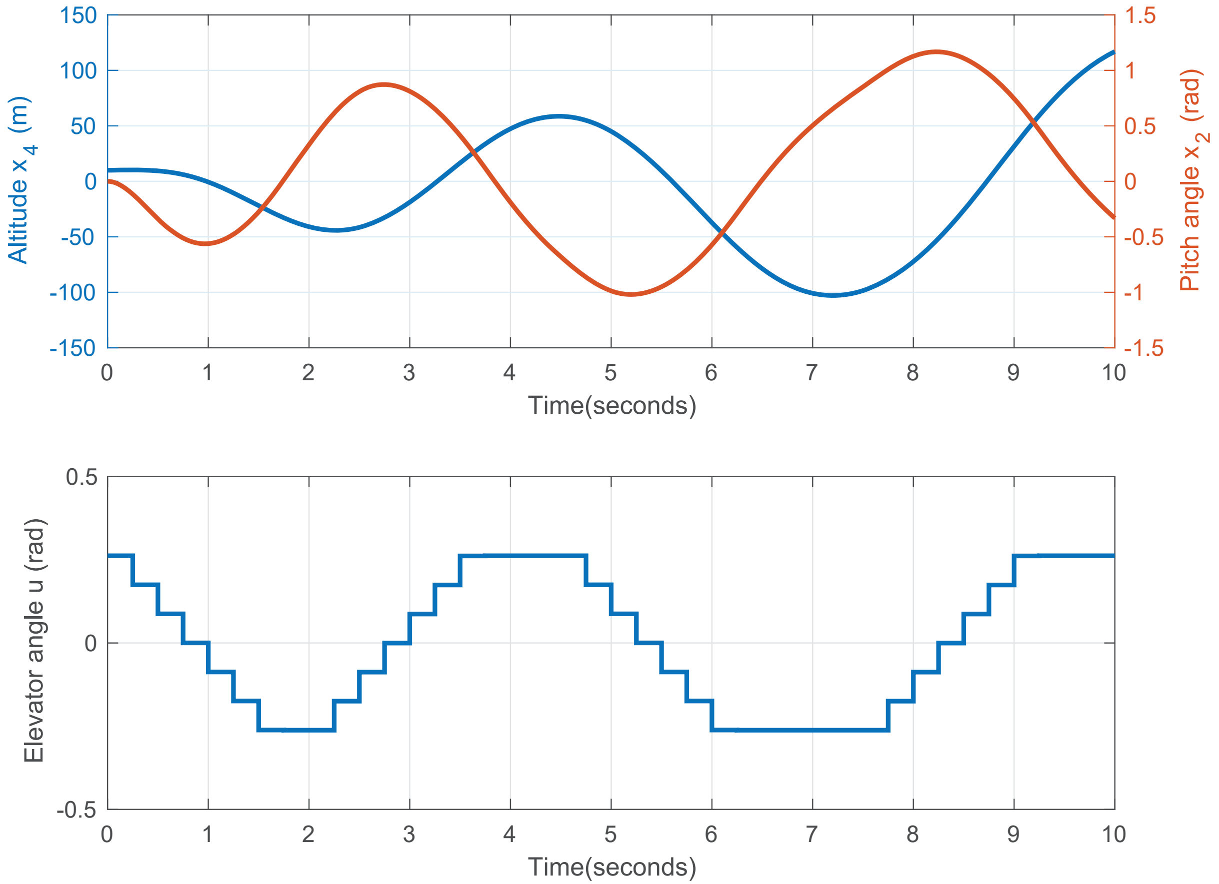
Problem parameters:
Sampling time 0.25sec,
$Q = I$, $R = 10$

Closed-loop system is unstable

Applying LQR control and saturating the controller can lead to instability!

Linear MPC Example: MPC with Input Bound Constraints

MPC controller with input constraints $|u_i| \leq 0.262$



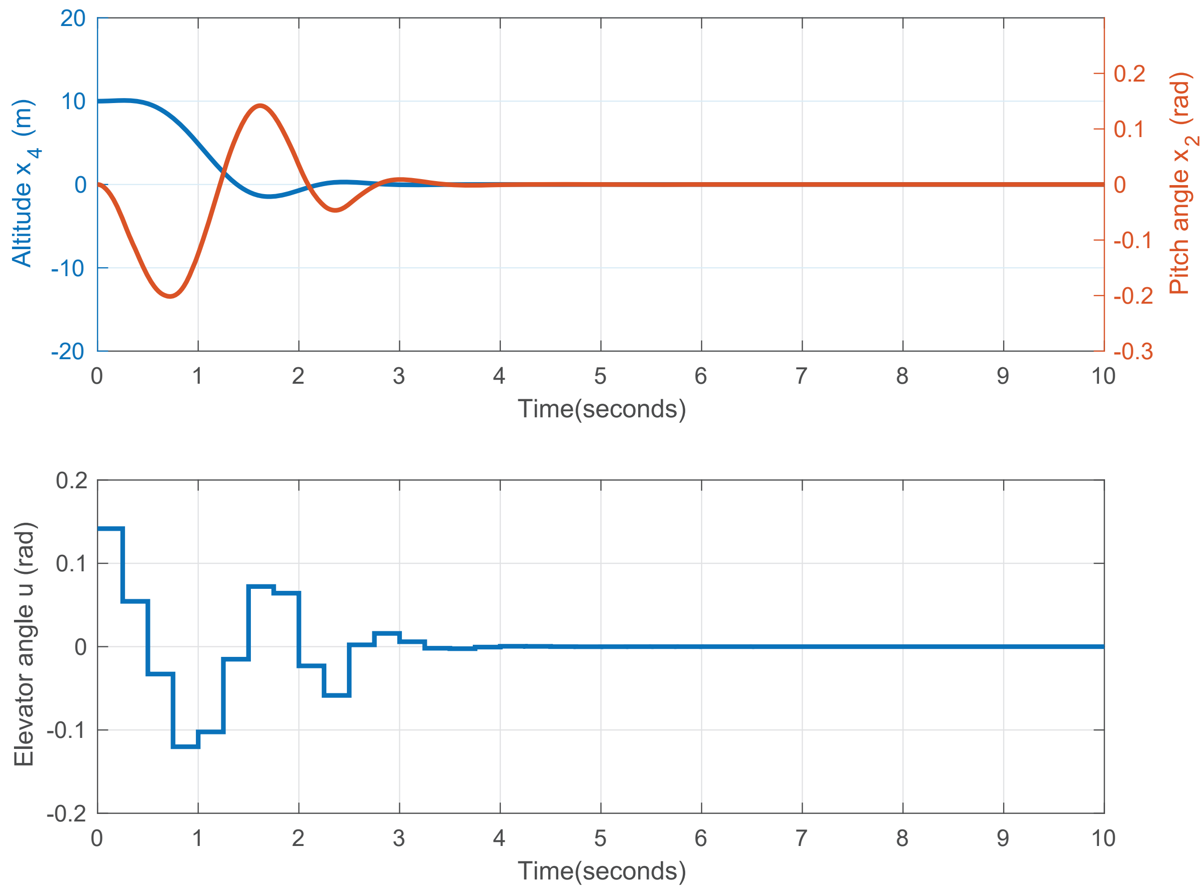
Problem parameters:
Sampling time $0.25sec$,
$Q = I$, $R = 10$, $N = 10$

The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

$\Rightarrow$System does not converge to desired steady-state but to a limit cycle

Linear MPC Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$, approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$



Problem parameters:
Sampling time $0.25 sec$,
$Q = I$, $R = 10$, $N = 10$

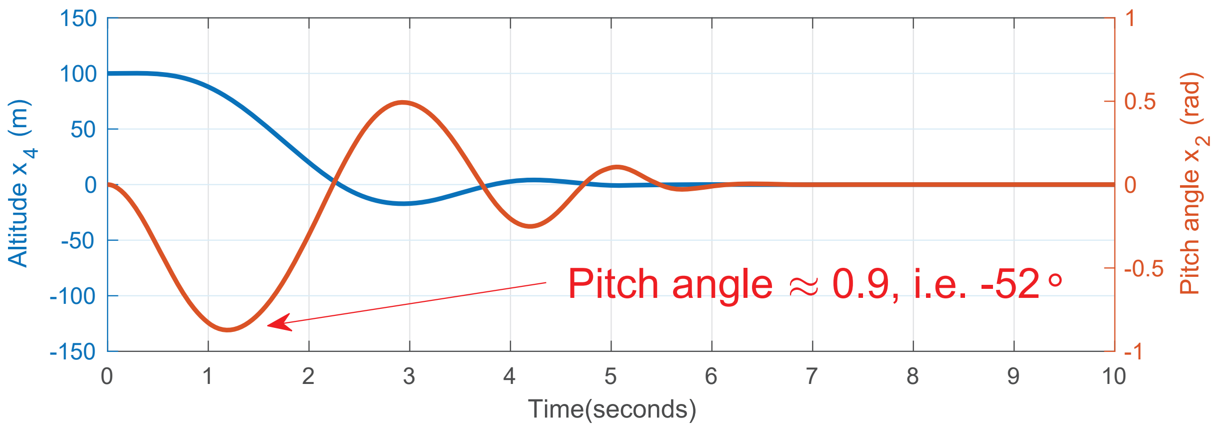
The MPC controller considers all constraints on the actuator

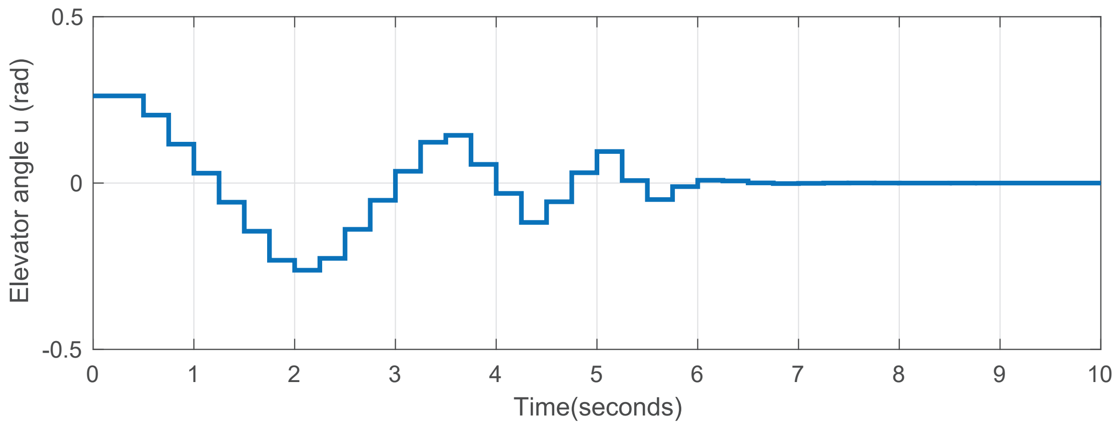
Closed-loop system is stable

Efficient use of the control authority

Linear MPC Example: Inclusion of State Constraints

MPC controller with input constraints $|u_i| \le 0.262$ and rate constraints $|\dot{u}_i| \le 0.349$, approximated by $|u_k - u_{k-1}| \le 0.349 T_s$





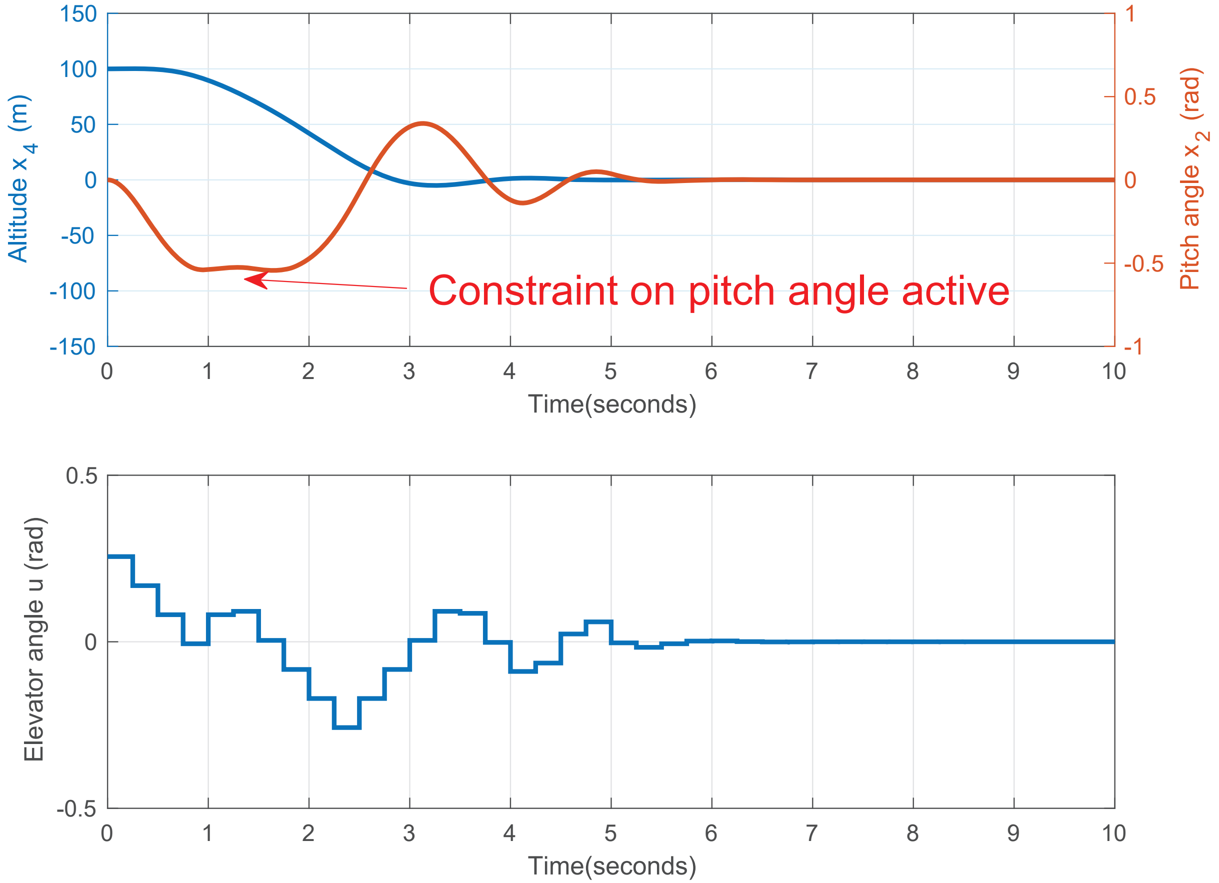
Problem parameters:
Sampling time $0.25sec$,
$Q = I$, $R = 10$, $N = 10$

Increase step :
At time $t = 0$ the plane is flying with a deviation of $100m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 100]$

- Pitch angle too large during transient

Linear MPC Example: Inclusion of State Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$, approximated by $|u_k - u_{k-1}| \leq 0.349 T_s$

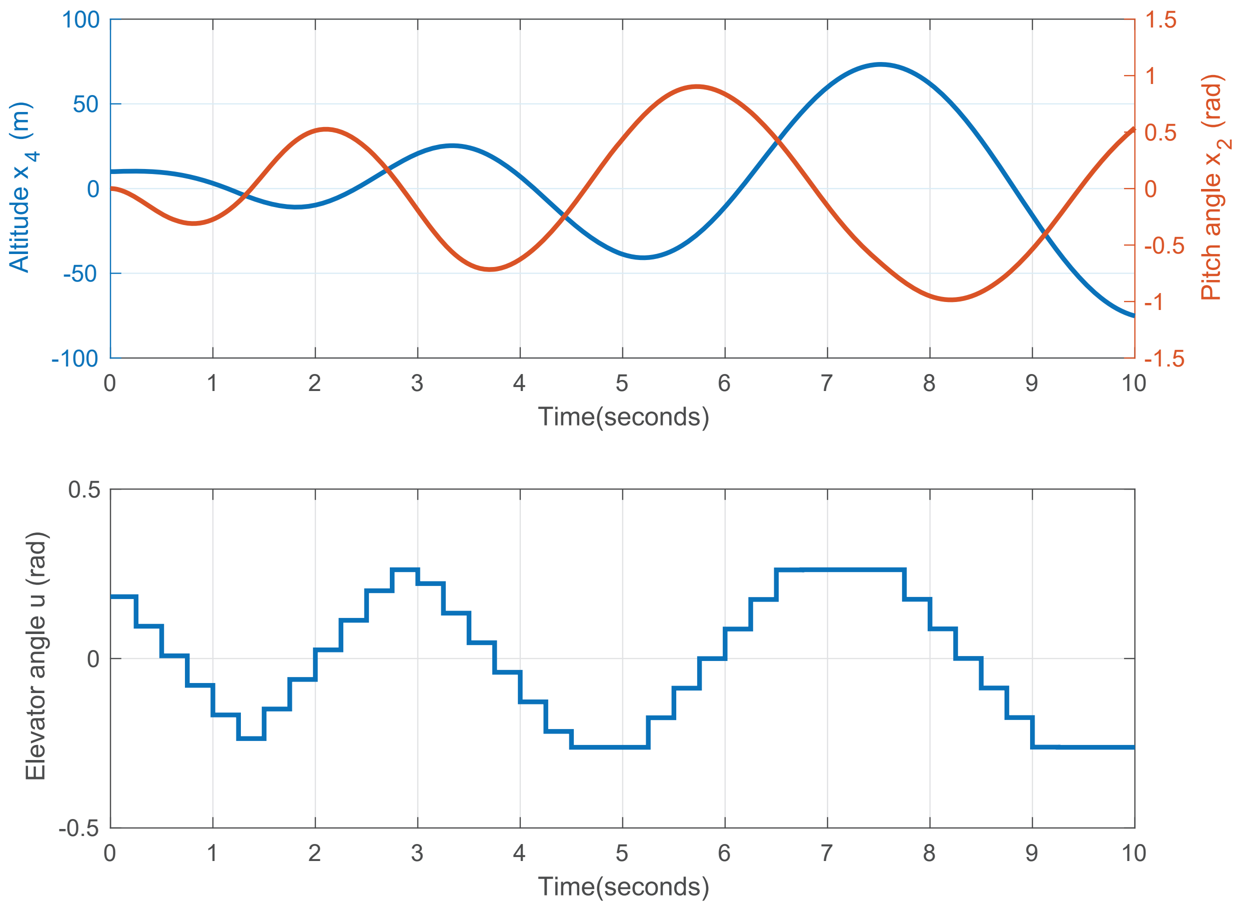


Problem parameters:
Sampling time $0.25 sec$,
$Q = I$, $R = 10$, $N = 10$

Add state constraints for passenger comfort:
$|x_2| \leq 0.650$

Linear MPC Example: Short horizon (1/2)

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$, approximated by $|u_k - u_{k-1}| \leq 0.349T_s$

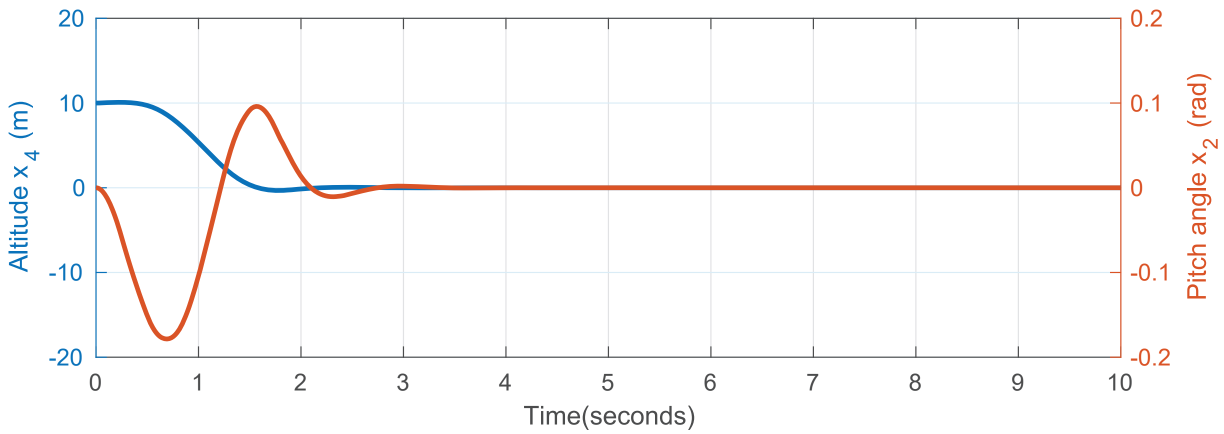


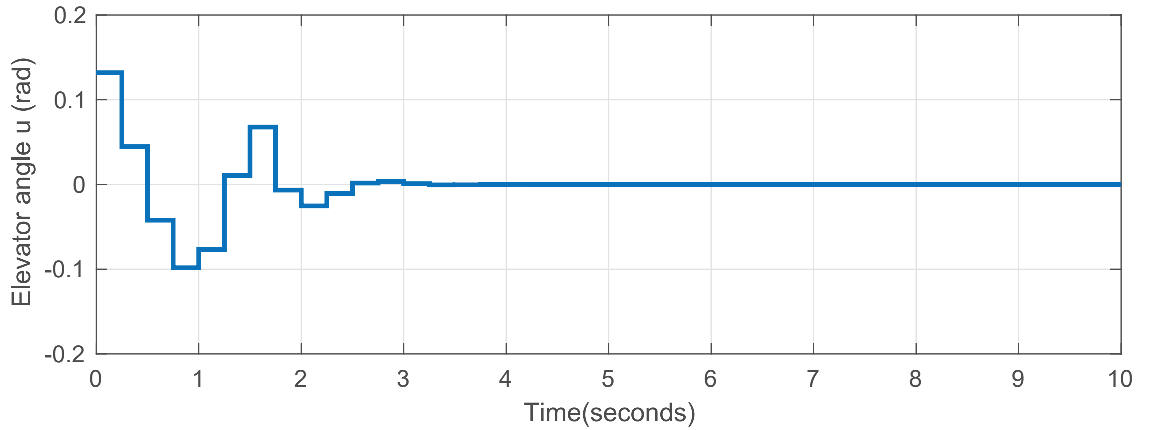
Problem parameters:
Sampling time $0.25sec$,
$Q = I$, $R = 10$, $N = 6$

Decrease in the prediction horizon causes loss of the stability properties

Linear MPC Example: Short horizon (2/2)

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$, approximated by $|u_k - u_{k-1}| \leq 0.349T_s$





Problem parameters:
Sampling time $0.25sec$,
$Q = I$, $R = 10$, $N = 6$

Inclusion of terminal cost and constraint provides stability

Conclusion

- Introduced general model predictive control formulation:
  - At each time t, update current state $x(t)$ and solve an optimal control problem over a finite look ahead horizon with $x(t)$ as the initial state
  - Apply the first step of the obtained optimal control sequence to the system
  - $t \leftarrow t+1$ and repeat the above steps
- For linear systems: using 2-norm, 1-norm, or ∞-norm cost functions all lead to tractable solutions for the online optimization problem
- Powerful toolbox: Multi-parametric toolbox (MPT) ←

Manfred
ETH: Morari
Bemprad
Borrelli
Berkelly
WBC

MPC toolbox ✗