MEE5114 Advanced Control for Robotics

Lecture 12: Basics of Feedback Linearization

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Outline

- Motivating Examples

- Input-Output Linearization

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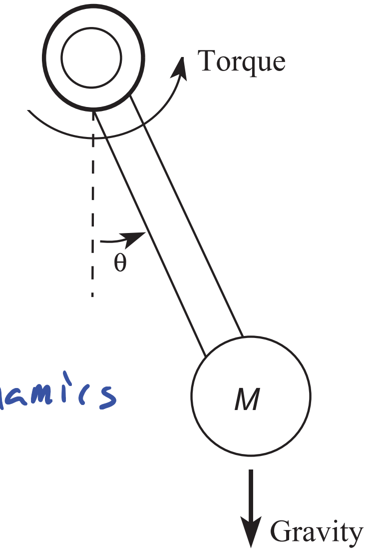
Motivating Example I

- Consider pendulum dynamics equation:
  $\ddot{\theta} = -a\sin(\theta) - b\dot{\theta} + c\tau$
  - $a = g/l$, $b = k/m$, $c = 1/ml^2$ are constants
  - $\tau > 0$ is the applied torque
  - State-space form:



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin(x_1) - bx_2 + c\underline{u} \\ y = \underline{x_1} \end{cases}$$

nonlinear dynamics

$u \rightarrow$ Pendulum $\rightarrow y$

  - Objective: regulate $y(t)$ to zero

· take derivative of y: $\dot{y} = \dot{x}_1 = x_2$, $\ddot{y} = \dot{x}_2 = -a\sin x_1 - bx_2 + cu$

– suppose $c \neq 0$, we can arbitrarily control $\ddot{y}$

e.g. choose $\boxed{u = a\sin x_1 + b x_2 + v} \Rightarrow \ddot{y} = v$ ← double integrator

let $\xi_1 = y$

$\xi_2 = \dot{\xi}_1 = \dot{y}$

Motivating Example I

$$\Downarrow$$

$$\Leftarrow \boxed{\begin{cases}\dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = v\end{cases}, \quad y = \xi_1}$$

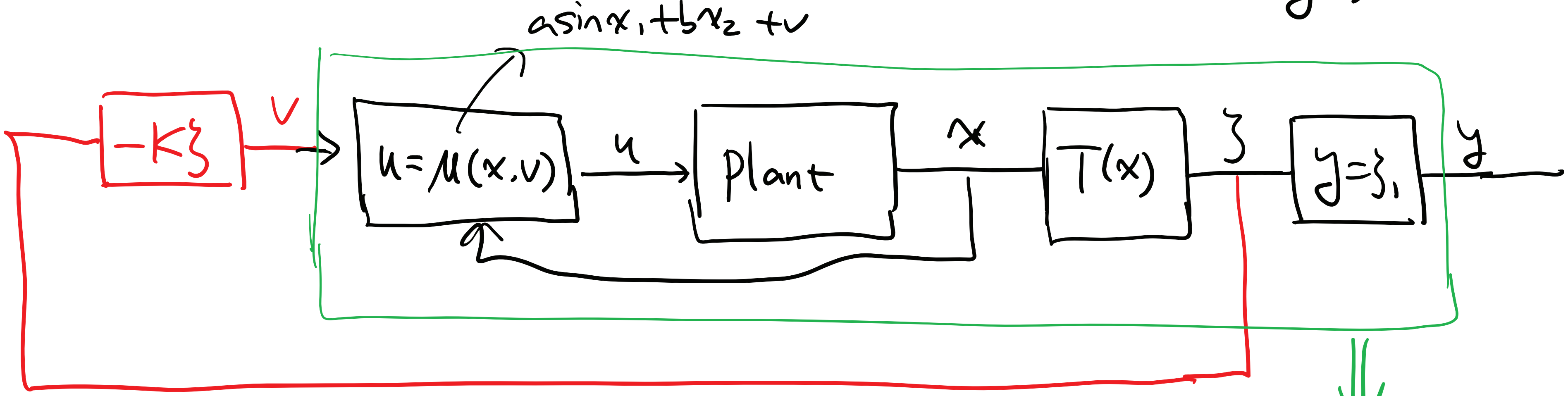
- Design linear control

$$v = -k_1 \xi_1 - k_2 \xi_2$$

$$\Rightarrow \begin{bmatrix}\dot{\xi}_1 \\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -k_1 & -k_2\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} \Rightarrow \begin{matrix}k_1 > 0 \\ k_2 > 0\end{matrix} \Rightarrow \xi_1, \xi_2 \to 0$$

$$\Rightarrow y \to 0$$

- Overall stabilizing controller:



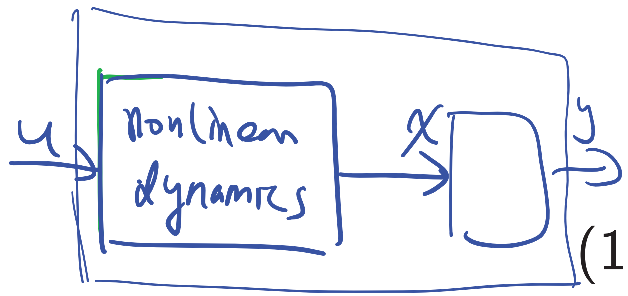
$$\Downarrow$$

linear
double integrator

Motivating Example II (1/3)

Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = a x_2^3 + u \\ \dot{x}_2 = u \\ y = x_1 \end{cases} \tag{1}$$



- Objective: make $y(t)$ track $y_d(t)$, i.e., make $e(t) = y(t) - y_d(t)$ go to zero.
- To reveal direct relationship between y and u, taking derivative of y $\dot{y} = x_2^3 + u$.
- Above equation indicates that one can directly control the derivative of y, and hence the derivative of e.

choose: $u = \mu(x, v) = -x_2^3 + v \Rightarrow \dot{y} = v$

choose $v = \dot{y}_d - k(y - y_d)$

$\Rightarrow \dot{y} - \dot{y}_d + k(y - y_d) = 0$

Motivating Example II (2/3) $\Rightarrow \dot{e} + ke = 0$

- We have $e \to 0$, i.e., $y(t) = x_1(t) \to y_d(t)$. What about $x_2(t)$?

$u = \mu(x, v) = -x_2^3 + \dot{y}_d - ke$

- Plugging in, we have

internal dynamics

$$\underbrace{\dot{x}_2 + ax_2^3}_{\dot{x}_2 = -ax_2^3 + \dot{y}_d - e(t)} = \dot{y}_d - e(t) \tag{2}$$

- μ makes $e(t) \to 0$, but it may result in diverging x_2.

Motivating Example II (3/3)

- System (1) has dimension $n = 2$.
  - Linearized input/output relation: $\dot{e} + e = 0$. The order of this I/O dynamics is called the *relative degree (i.e. here $r = 1$)*
  - *The remaining dynamics (2) has dimension: $n - r = 1$. This dynamics is called* internal dynamics

Summary Based on Motivating Examples

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Outline

- Motivating Examples

- Input-Output Linearization

Relative Degree

- Consider single-input-single-output control affine system:

single-input single output (SISO)

$$\begin{cases} \dot{x} = \underbrace{f(x) + g(x)u}_{\text{Input affine}} & u \in \mathbb{R} \\ y = \underbrace{h(x)}_{\text{No direct feedthrough}} \end{cases} \tag{3}$$

- **Relative Degree**: Roughly speaking, relative degree is the number of times we need to take the time derivatives of the output to see the input:
  - $\dot{y} = \left(\frac{\partial h}{\partial x}\right)^T f(x) + \left(\frac{\partial h}{\partial x}\right)^T g(x)u \triangleq L_f h(x) + \boxed{L_g h(x)}u$

    $\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \vdots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$ $\quad f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$

  - If $L_g h(x) \neq 0$ in an open set containing the equilibrium, then relative degree is 1. If not, continue taking derivative:

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u$$

  - If $L_g L_f h(x) \neq 0$, then relative degree is 2, otherwise continue ...

Relative Degree (Continued)

- **Definition (Relative Degree):** System (3) has relative degree r if, in a neighborhood of the equilibrium,

$$L_g L_f^{i-1} h(x) = 0, i = 1, \ldots, r-1, \text{ and } L_g L_f^{r-1} \neq 0$$

- Example 1: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^3 + u$, $y = x_1$

$$\dot{y} = \dot{x}_1 = x_2 + 0 \cdot u \qquad \ddot{y} = \dot{x}_2 = -x_1^3 + \underbrace{u}_{L_g(L_f h)}$$

$\Rightarrow RD = 2$

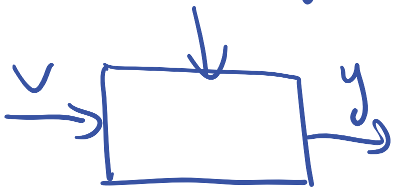
Let $u = x_1^3 + v \Rightarrow \ddot{y} = v$

Let $\xi_1 = y$, $\xi_2 = \dot{y}$ $\Rightarrow$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ v \end{bmatrix}$$

$$y = \xi_1$$

double integrater



Relative Degree (Continued)

- Example 2: $\dot{x} = Ax + Bu, \quad y = Cx$

- Example 3: $\dot{x}_1 = x_2 + x_3^3$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$, $y = x_1$

$$\dot{y} = x_2 + x_3^3 \quad \Rightarrow \quad \ddot{y} = \dot{x}_2 + 3x_3^2 \cdot \dot{x}_3 = x_3 + 3x_3^2 \cdot u$$

can we control $\ddot{y}$?

If $x_3 \neq 0$, we can control $\ddot{y}$

$x_3 = 0$, we can't.

System does not have well-defined RD around $x_3 = 0$

but system can be viewed as "double-integrator" for $x_3 \neq 0$

Input-Output Linearization

If system (3) has a well defined relative degree $r \geq n$, then it is input-output linearizable

- $y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0}u$ → $\mathbb{R}^1$

  $\neq 0$ { globally or locally around some point $\hat{x}$

- Apply feedback: $u = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v \right) \Rightarrow y^{(r)} = v$

  ⇓ directly control $y^{(r)}$

- Integrator chain:

If $r \geq n$. system (fully) Input-Output linearizable

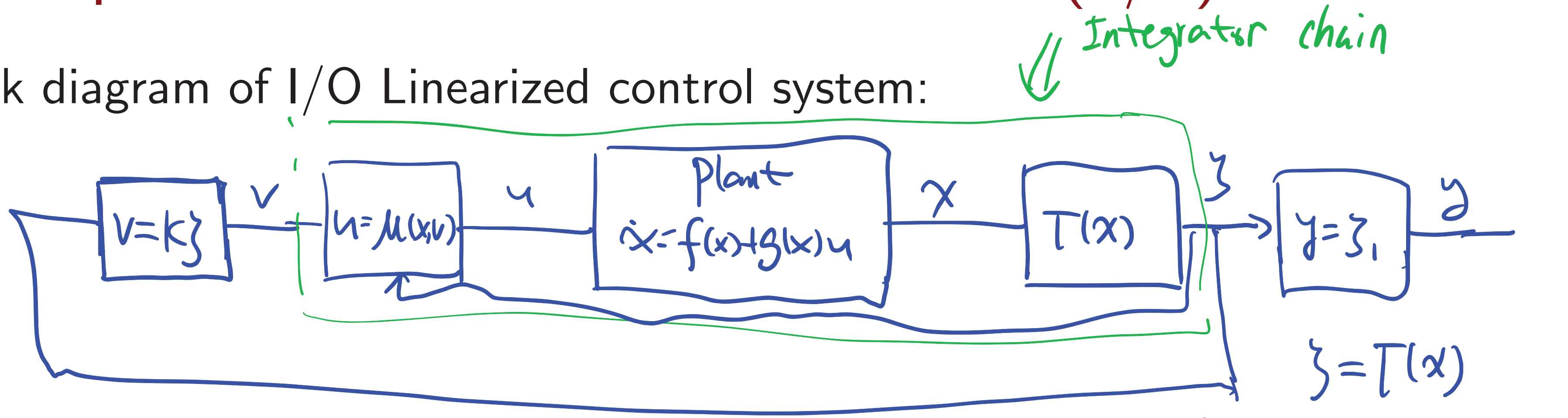
$$\begin{cases} \dot{\zeta}_1 = \zeta_2 \\ \vdots \\ \dot{\zeta}_r = v \end{cases} \tag{4}$$

- $r < n$, partially I/O linearizable

where $\zeta_1 = y = h(x)$, $\zeta_2 = \dot{y} = L_f h(x)$, $\ldots$, $\zeta_r = y^{(r)} = L_f^{r-1} h(x)$.

Input-Output Linearization: Normal Form (1/2)

- Block diagram of I/O Linearized control system:



If $r=n$, $T:\mathbb{R}^n \to \mathbb{R}^n$

- Sys (4) if r-dim, if $r < n$, then there will be remaining internal dynamics

- Let $z \in \mathbb{R}^{n-r}$ be the state variable of the internal dynamics, feedback linearization is essentially representing the dynamics of x using new coordinate $(\underbrace{z}_{\mathbb{R}^{n-r}}, \underbrace{\zeta}_{\mathbb{R}^r})$, which is function of x:

$$(z(x), \zeta(x)) = \phi(x)$$

- For valid state transformation, ϕ and ϕ^{-1} need to be continuously differentiable, such a ϕ mapping is called a diffeomorphism

Input-Output Linearization: Normal Form (2/2)

- **Theorem**: Suppose system (3) has a well-defined relative degree $r \leq n$, then there exists a diffeomorphism $\phi(x) = (z, \zeta)$ with $z \in \mathbb{R}^{n-r}$ and $\zeta \in \mathbb{R}^r$, that transforms the system to the form:

$$\begin{cases} \dot{z} = f_0(z, \zeta) \quad \mathbb{R}^{n-r} \\ \dot{\zeta}_1 = \zeta_2 \\ \vdots \\ \dot{\zeta}_r = \beta(z,\zeta) + \alpha(z,\zeta)u \\ y = \zeta_1 \end{cases}$$

- $\zeta = [h(x) \quad L_f h(x) \quad \cdots \quad L_f^{r-1} h(x)]^T$

- $z_1, \ldots, z_{n-r}$ are $n - r$ independent variables such that $\dot{z}$ does not contain u

- ϕ can be found by solving a set of PDEs (hard in general).

Internal Dynamics and Zero Dynamics

- Dynamics $\dot{z} = f_0(z, \zeta)$ is $(n - r)$-dimensional and is called the internal dynamics

$v \to \boxed{} \to \xi$ integrator chain

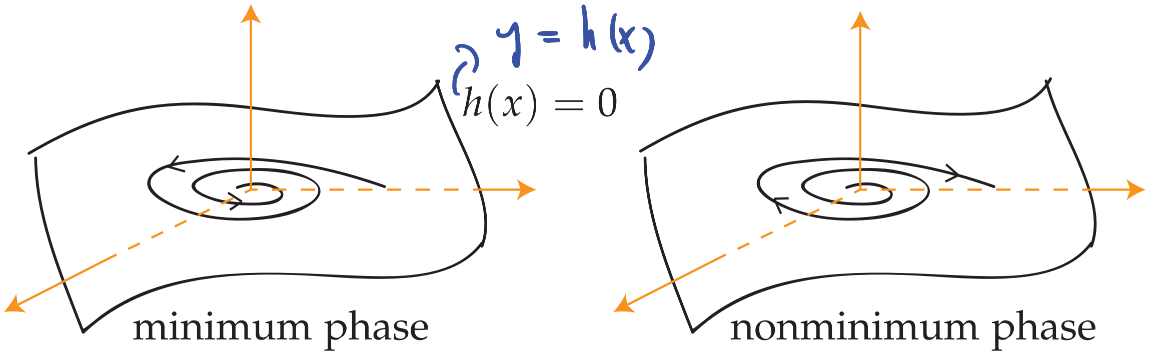
- Internal dynamics should be viewed as dynamics of the internal state z with ζ being an external input

- Typically, we want $\zeta \to 0$, thus it is important to study dynamics

$$\dot{z} = f_0(z, 0)$$

which is called the **zero dynamics** $\quad \xi_1 = 0,\ \dot{\xi}_1 = \xi_2 = 0 \ldots \Rightarrow y = 0$

- If the origin of the zero dynamics is asymp. stable $\Rightarrow$ minimum-phase system; otherwise $\Rightarrow$ nonminimum phase



I/O Linearization in Normal Form

- I/O Linearizing controller in new coordinate:

$$\begin{cases} u = \frac{1}{\alpha(z,\zeta)}\left(-\beta(z,\zeta) + v\right) \\ v = -k_1\zeta_1 - \cdots - k_r\zeta_r \end{cases} \tag{5}$$

- **Theorem** If $z = 0$ is locally expeontially stable for $\dot{z} = f_0(z, 0)$, then controller (5) locally exponentially stabilizes $x = 0$ for system (3)

Roughly speaking, we can ignore the internal dynamics under this condition

- **Theorem** Global asymp stability can be guaranteed if $\dot{z} = f_0(z, \zeta)$ is ISS with respect to input ζ

$\underbrace{\text{Input-to-state}}_{\text{ISS}}$ stable

$\dot{x} = f(x, u)$ is ISS

if $\|x(t)\| \le e^{-ct} \cdot \|x(0)\| + c_2 \| \sup_{0 \le \tau \le t} |u(\tau)| \|$

Feedback Linearization for Tracking

- Suppose we want the output y of sys (3) to track a reference signal $y_d(t)$

- Choose v as
$$v = -k_1(\zeta_1 - y_d) - \cdots - k_r(\zeta_r - \dot{y}_d^{(r)})$$
- Let $e_i = \zeta_i - y_d^{(i)}$, $i = 0, \ldots r-1$

- Then we know $e(t) \to 0$, i.e., $\|y(t) - y_d(t)\| \to 0$

- If all derivatives of y_d are bounded, then ζ is bounded. If zero dynamics $\dot{z} = f_0(z, \zeta)$ is ISS (Input-to-State Stable), then $z(t)$ is also bounded. All internal signals are bounded.

More About Zero Dynamics

- Set $y = 0, \dot{y} = 0, \ldots, y^{(r-1)} = 0$ and substitute (4) with $v = 0$, i.e.,

$$u^* = \frac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x)\right)$$

- The remaining dynamical equations describe the zero dynamics
- Example: Cart Pole

$$\begin{cases} \ddot{y} = \frac{1}{\frac{M}{m}+\sin^2(\theta)}\left(\frac{u}{m} + \dot{\theta}^2 l \sin(\theta) - g\sin(\theta)\cos(\theta)\right) \\ \ddot{\theta} = \frac{1}{l(\frac{M}{m}+\sin^2(\theta))}\left(-\frac{u}{m}\cos(\theta) - \dot{\theta}^2 l \cos(\theta)\sin(\theta) + \frac{M+m}{m} g \sin(\theta)\right) \end{cases}$$

state space: 4 dim: $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \theta$, $x_4 = \dot{\theta}$

Dynamics: $\dot{y} = x_2$, $\ddot{y} =$ we see "u" in $\ddot{y}$

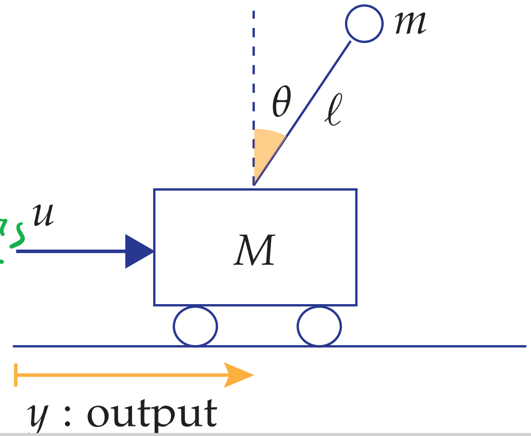
$\Rightarrow$ RD = 2

Select: $u^* = \underline{-m\left(\dot{\theta}^2 l \sin\theta - g \sin\theta\cos\theta\right)} + m\left(\frac{M}{m} + \sin^2\theta\right) v$

To find zero dynamics: $\underline{y = 0, \dot{y} = 0}$

let $\underline{v = 0}$, $u = u^*$ $\Rightarrow$ after algebra: $\boxed{\ddot{\theta} = \frac{g}{l}\sin\theta}$ zero-dynamics

unstable



Summary of Input-Output Linearization

at $(\theta = 0, \dot{\theta} = 0)$

- Differentiate the output y until the input u appears

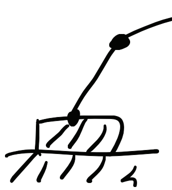
- Choose u to cancel the nonlinearities and guarantee tracking convergence

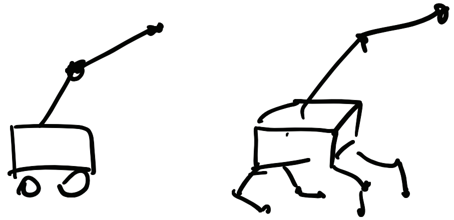
- Study the stability of the internal dynamics

  - Minimum-phase

  - Non-minimum-phase

More Discussions

-  fixed base system

- Floating base system: 

$\Downarrow$

- overall configuration : $q = [\underbrace{q_B}_{\text{float-base}} \;\; q_J] \in \mathbb{R}^{n+6}$, $q_B \in \mathbb{R}^6$, $q_J \in \mathbb{R}^n$

- spatial velocity $\dot{q}_B = \mathcal{V}_B$

$$\Rightarrow \dot{q} = \begin{bmatrix} \mathcal{V}_B \\ \dot{q}_J \end{bmatrix}$$

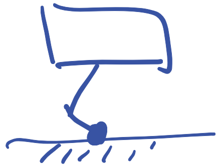
$$\Rightarrow \quad M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g = \tau + J^T \mathcal{F}_{ext}$$

More Discussions

- $$\Rightarrow \underbrace{\begin{bmatrix} M_{BB} & M_{BJ} \\ M_{JB} & M_J \end{bmatrix}}\begin{bmatrix}\ddot{q}_B\\ \ddot{q}_J\end{bmatrix} + \begin{bmatrix} C_B \\ C_J\end{bmatrix}\cdot\begin{bmatrix}\dot{q}_B\\ \dot{q}_J\end{bmatrix} + \begin{bmatrix}\tau_{gB}\\ \tau_{gJ}\end{bmatrix} = \begin{bmatrix} 0 \in \mathbb{R}^{6\times 1} \\ \tau_J\end{bmatrix} + J^T(q)\,\underset{\sim}{F_{ext}}$$

$$= \underbrace{\begin{array}{c} 6\times n \\ \begin{bmatrix} 0 \\ I \end{bmatrix} \\ n\times n \end{array}}_{S\ :\ \text{selection}} \tau_J$$

– point contact



$$p_c = \underbrace{\phi(q)}_{\text{forward kinematics}} \equiv c$$

$$\Rightarrow \quad \dot{\phi}(q) = 0 \quad \Rightarrow \quad \underbrace{\left(\frac{\partial \phi}{\partial q}\right)}_{J_c(q)\ :\ \text{Analytical Jacobian}} \cdot \dot{q} = 0 \quad \Rightarrow \quad \ddot{\phi}(q) = 0$$

$J_c(q)$: Analytical Jacobian

$$\Rightarrow \quad J_c \ddot{q} + \dot{J}_c \dot{q} = 0$$

$\Rightarrow$ ~~beca~~ Dynamics :

$$\begin{cases} M\ddot{q} + C\dot{q} + \tau_g = S\tau_J + J_c^T f_c \\ J_c\ddot{q} + \dot{J}_c \dot{q} = 0 \end{cases}$$

More Discussions

$$\Rightarrow f_c = \underbrace{(J_c^T)^{\dagger} M J_c^{-1}}_{(J_c M^{-1} J_c^T)^{-1} \text{ Contact inertia}} \left(- J_c^T \dot{q} - J_c M^{-1} (S \tau_J - c \dot{q} - \tau_g) \right)$$

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