MEE5114 Advanced Control for Robotics

Lecture 2: Rigid Body Configuration and Velocity

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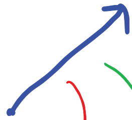
Outline

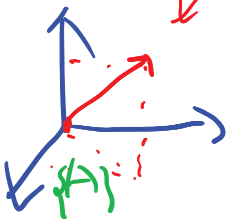
- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

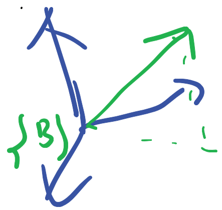
Free Vector

- **Free Vector**: geometric quantity with length and direction



- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented by its coordinates v in the reference frame.





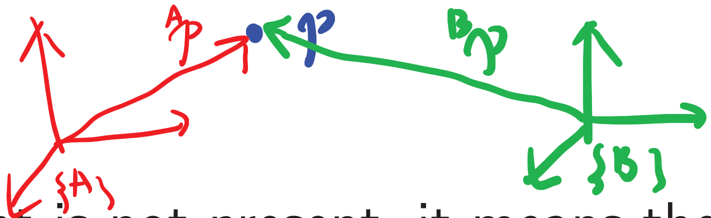
- v denotes the physical quantity while ${}^{A}v$ denote its coordinate wrt frame $\{A\}$.

Point

- **Point**: p denotes a point in the physical space

- A point p can be represented by a vector from frame origin to p

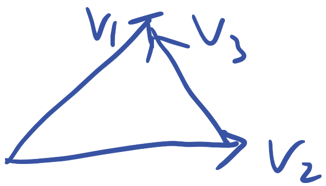
- ${}^{A}p$ denotes the coordinate of a point p wrt frame {A}



- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector where reference frame is clear from the context.

think in "coordinate-free" way whenever possible

e.g



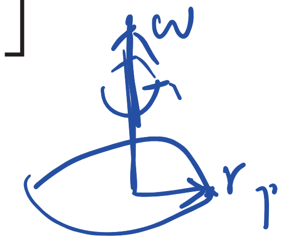
$V_3 = V_1 - V_2 \leftarrow$ coordinate free

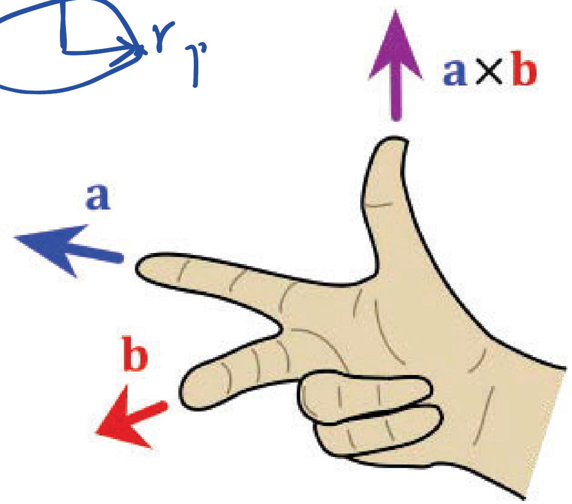
$\Rightarrow$ ${}^{A}V_3 = {}^{A}V_1 - {}^{A}V_2$, ${}^{B}V_3 = {}^{B}V_1 - {}^{B}V_2$

Cross Product

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{1}$$





Properties:

- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$

linear velocity

$v_p = w \times r_p$

moment

$m_p = r_p \times f$

Skew symmetric representation

- It can be directly verified from definition that $a \times b = [a]b$, where

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

- $[a] = -[a]^T$ (called skew symmetric)

- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$a \times b = [a] \cdot b$$

$$= \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

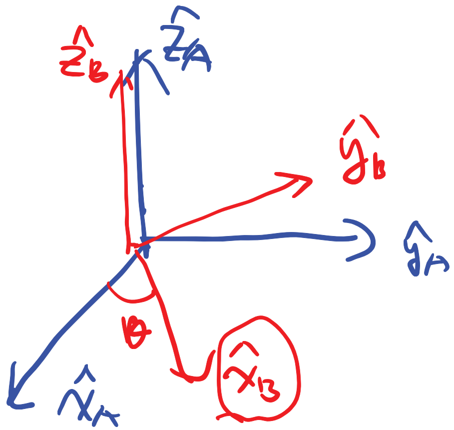
$$= \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$

- **Rotation Matrix**: specifies orientation of one frame relative to another

$$^{A}R_{B} \triangleq \left[\begin{array}{c|c|c} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} & {}^{A}\hat{z}_{B} \end{array}\right]$$



$$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \cdots \quad (1)$$

- A valid rotation matrix R satisfies: (i) $R^T R = I$; (ii) $\det(R) = 1$

$$\begin{bmatrix} {}^{A}x_B^T \\ {}^{A}y_B^T \\ {}^{A}z_B^T \end{bmatrix} \begin{bmatrix} {}^{A}x_B & {}^{A}y_B & {}^{A}z_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(R) = {}^{A}x_B^T \left({}^{A}y_B \times {}^{A}z_B\right) = 1$$

Special Orthogonal Group

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : R^T R = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, $\forall a, b, c \in G$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Use of Rotation Matrix

By definition, ${}^{A}R_B$

- Representing an orientation of frame {B} relative to {A}

  Given a vector $\underline{v}$, it's coordinate in {A},

- Changing the reference frame: {B} are related by

$$\boxed{\underbrace{{}^{A}v = {}^{A}R_B {}^{B}v}_{\text{the same vector}}}$$

- Rotating a vector or a frame (to be discussed more later)

  Theorem (Euler): Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$

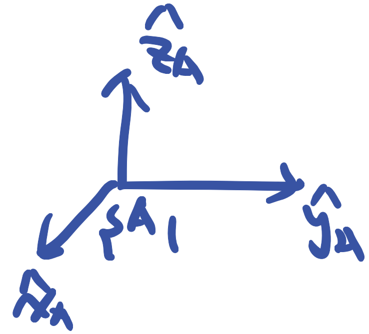
$$R = \boxed{\mathrm{Rot}(\hat{\omega}, \theta)} = e^{At}$$

$A = [\hat{\omega}]$

$\theta = t$

- "coordinate free" proof

$$\boxed{{}^{A}V = {}^{A}R_{B}\, {}^{B}V}$$



same physical vector v

Its coordinate in $\{A\}$ frame is ${}^{A}V = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ or

$$\Rightarrow \quad v = \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A$$

Its coordinate in $\{B\}$ frame is ${}^{B}V = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

$$\Rightarrow \quad v = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B$$

"coordinate free" statement

$$\Rightarrow \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B$$

$\Downarrow$

choose "{A} frame to express this "physics"

$$\Rightarrow \underbrace{\alpha_1\, {}^{A}\hat{x}_A + \alpha_2\, {}^{A}\hat{y}_A + \alpha_3\, {}^{A}\hat{z}_A}_{} = \beta_1\, {}^{A}\hat{x}_B + \beta_2\, {}^{A}\hat{y}_B + \beta_3\, {}^{A}\hat{z}_B$$

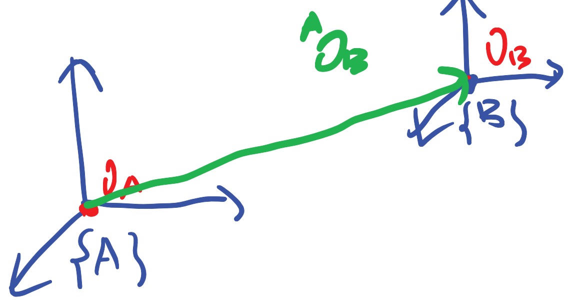
$$\Rightarrow \underbrace{\begin{bmatrix} {}^A\hat{x}_A & {}^A\hat{y}_A & {}^A\hat{z}_A \end{bmatrix}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}} = {}^A R_B \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\Rightarrow \quad {}^A V = {}^A R_B \, {}^B V$$

Rigid Body Configuration

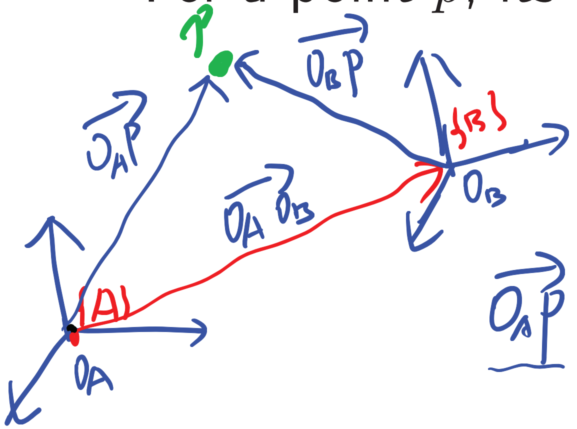
- Given two coordinate frames $\{A\}$ and $\{B\}$, the configuration of B relative to A is determined by
  - ${}^AR_B$ and ${}^Ao_B$



- For a (free) vector r, its coordinates ${}^Ar$ and ${}^Br$ are related by:

$$ {}^Ar = {}^AR_B\,{}^Br $$

- For a point p, its coordinates ${}^Ap$ and ${}^Bp$ are related by:



"coordinate free"

$$ \boxed{{}^Ap = {}^Ao_B + {}^AR_B\,{}^Bp} $$

$$ \overrightarrow{O_AP} = \overrightarrow{O_AO_B} + \overrightarrow{O_BP} \Longrightarrow {}^A\overrightarrow{O_AP} = {}^A\overrightarrow{O_AO_B} + {}^A\overrightarrow{O_BP} $$

By definition

choose "A" frame to express physics

Homogeneous Transformation Matrix

- **Homogeneous Transformation Matrix**: ${}^AT_B$

$${}^Ap = {}^AO_B + {}^AR_B\,{}^Bp \iff \underbrace{\begin{bmatrix} {}^Ap \\ 1 \end{bmatrix}}_{4\times 1} \in \mathbb{R}^3 = \underbrace{\begin{bmatrix} \overset{3\times 3}{{}^AR_B} & \overset{3\times 1}{{}^AO_B} \\ \underbrace{0}_{1\times 3} & \underbrace{1}_{1\times 1} \end{bmatrix}}_{4\times 4} \underbrace{\begin{bmatrix} {}^Bp \\ 1 \end{bmatrix}}_{4\times 1}$$

$$\Rightarrow {}^AT_B \triangleq \begin{bmatrix} {}^AR_B & {}^AO_B \\ 0 & 1 \end{bmatrix}$$

$$T = (R, p)$$

- Homogeneous coordinates:

Given a point $p \in \mathbb{R}^3$, its homo coordinate is given by $\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}$

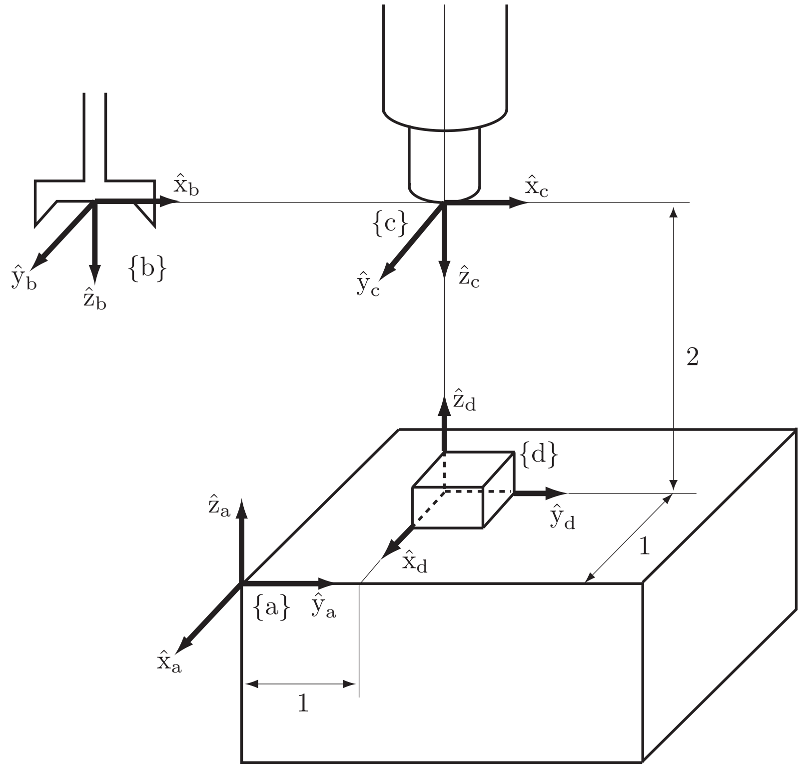
For vector v, its homogeneous coordinate is $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$v = p_1 - p_2$, $\tilde{v} = \tilde{p}_1 - \tilde{p}_2$

$$\Rightarrow {}^A\tilde{p} = {}^AT_B\,{}^B\tilde{p}$$

Example of Homogeneous Transformation Matrix

Fixed frame $\{a\}$; end effector frame $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$. Suppose $\|p_c - p_b\| = 4$

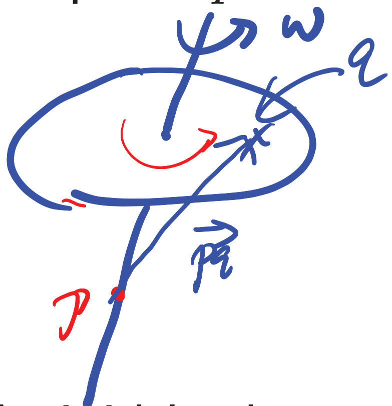


Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

Rigid Body Velocity (1/2)

- Consider a rigid body with angular velocity: ω (this is a free vector).
- Suppose the actual rotation axis passes through a point p; Let v_p be the velocity of the point p.



Question: A rigid body contains infinitely many points with different velocities. How to parameterize all of their velocities?

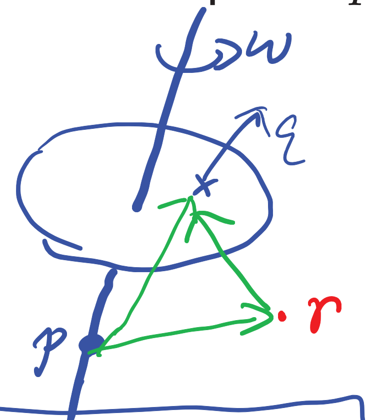
- Consider an arbitrary body-fixed point q (means that the point is rigidly attached to the body, and moves with the body), we have:

$$v_q = v_p + \omega \times (\overrightarrow{pq}) \tag{3}$$

- The velocity of an arbitrary body-fixed point depends only on (ω, v_p, p) and the location of the point q.

Rigid Body Velocity (2/2)

- Fact: The representation form (3) is independent of the reference point p.



- Consider an arbitrary point r in space
  - r may not be on the rotation axis
  - r may be a stationary point in space (does not move)
  - Let v_r be the **velocity of the body-fixed point currently coincides with** r
  - We still have: $v_q = v_r + \omega \times (\overrightarrow{rq})$

• $v_q = v_p + \omega \times (\overrightarrow{pq})$, $v_r = v_p + \omega \times (\overrightarrow{pr})$

$$= v_r - \omega \times (\overrightarrow{pr}) + \omega \times (\overrightarrow{pq})$$

$$= v_r + \omega \times (\overrightarrow{pq} - \overrightarrow{pr}) = v_r + \omega \times \overrightarrow{rq}$$

- The body can be regarded as translating with a linear velocity v_r, while rotating with angular velocity ω about an axis passing through r

Rigid Body Velocity: Spatial Velocity (Twist)

Roy Featherstone

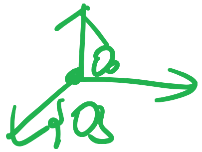
- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\omega, v_r)$
  - ω: angular velocity
  - v_r: velocity of the body-fixed point currently coincides with r
  - For any other body-fixed point q, its velocity is

$$\boxed{v_q = v_r + \omega \times (\overrightarrow{rq})}$$

- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point r

- A rigid body with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at v_r while rotating with angular velocity ω about an axis passing through r
  - This is just one way to interpret the motion.

Spatial Velocity Representation in a Reference Frame



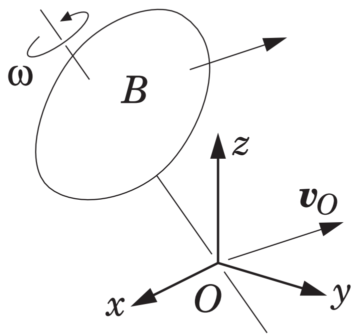
- Given frame $\{O\}$ and a spatial velocity $\mathcal{V}$

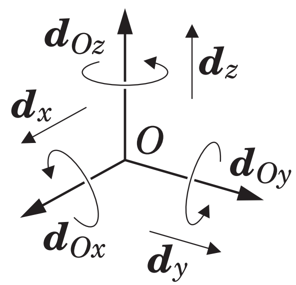
- Choose o (the origin of $\{O\}$) as the reference point to represent the rigid body velocity

- Coordinates for the $\mathcal{V}$ in $\{O\}$:

$$^{O}\mathcal{V}_o = (^{O}\omega, {}^{O}v_o)$$

- By default, we assume the origin of the frame is used as the reference point: $^{O}\mathcal{V} = {}^{O}\mathcal{V}_o$



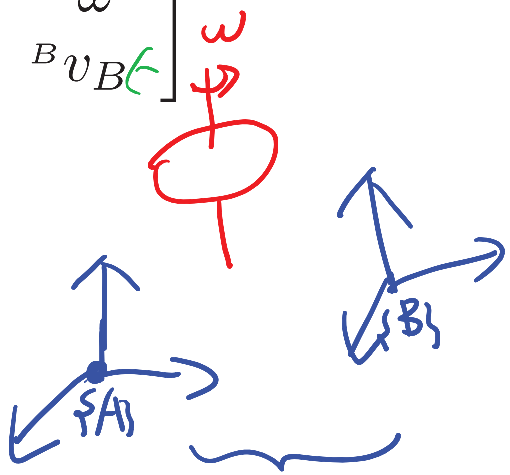


Change Reference Frame for Twist

$a\times b = -b\times a$

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames $\{A\}$ and $\{B\}$

$${}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix}$$



- They are related by ${}^A\mathcal{V} = {}^AX_B\,{}^B\mathcal{V}$

$\Rightarrow {}^A\omega = {}^AR_B\,{}^B\omega$

$= [{}^AR_B \ \vdots\ 0]\begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}$

"coordinate free" statement

v_A : body-fixed coincides with O_A

v_B : body-fixed velocity coincides with O_B

$\underbrace{{}^AT_B}{} = [{}^AR_B,\ {}^AO_B]$

$v_A = v_B + \omega \times (\overrightarrow{BA})$

choose $\{A\}$ to state: ${}^Av_A = {}^Av_B + {}^A\omega \times {}^A(\overrightarrow{BA}) = {}^AR_B\,{}^Bv_B + {}^A\omega \times (-{}^A\overrightarrow{AB})$

$= {}^AR_B\,\boxed{{}^Bv_B} + ({}^AO_B \times {}^AR_B\,\boxed{{}^B\omega})$

$[{}^AO_B]$

$= \Big[\underbrace{[{}^AO_B]\,{}^AR_B}_{3\times 3} \ \vdots\ \underbrace{{}^AR_B}_{3\times 3}\Big]\begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}$

- If configuration $\{B\}$ in $\{A\}$ is $T = (R, p)$, then

$$\underbrace{{}^A\mathcal{V}}_{6\times 1} = \underbrace{\begin{bmatrix} {}^AR_B & 0 \\ [{}^AO_B]\,{}^AR_B & {}^AR_B \end{bmatrix}}_{6\times 6} \underbrace{{}^B\mathcal{V}}_{6\times 1}$$

$${}^AX_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Example of Twist I

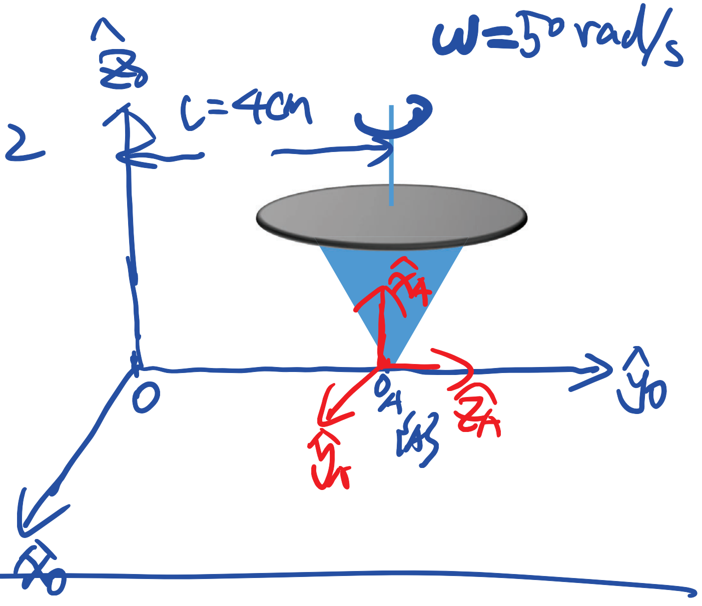
- Example I: what's the twist of the spinning top?

$\mathcal{V}$

choose "{0}"-frame: $${}^{0}\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50\ \text{rad/s} \\ \hline 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} {}^{0}\omega \\ {}^{0}v_{0} \end{pmatrix}$$

$0.04 \cdot 50 = 2$

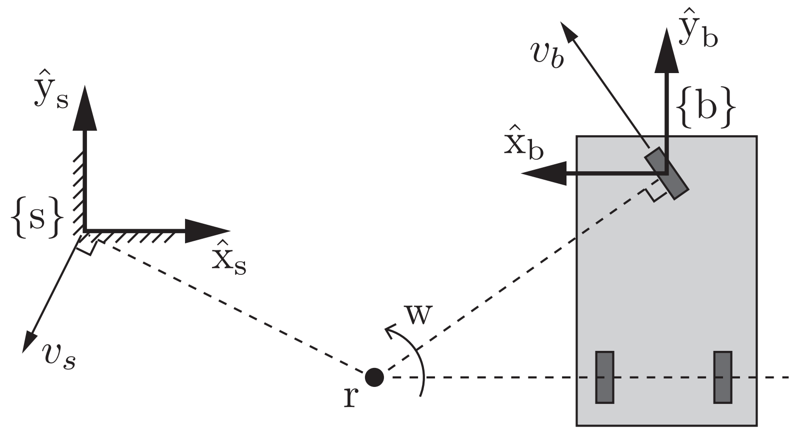


Choose {A}-frame $${}^{A}\mathcal{V}_{top} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^{A}\mathcal{V}_{top} = \begin{bmatrix} {}^{A}\omega \\ {}^{A}v_{O_A} \end{bmatrix}$$

Example of Twist II

- Example II:



$r_s = (2, -1, 0)$, $r_b = (2, -1.4, 0)$, w=2 rad/s

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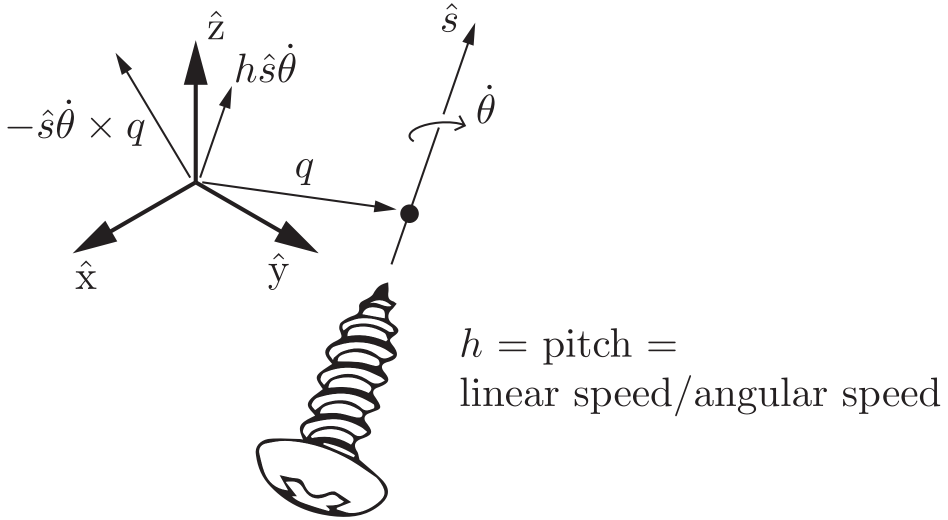
Screw Motion: Definition

- Rotating about an axis while also translating along the axis



- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis
- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

Why Screw Axis is Needed?

- Consider a rigid body with twist $\mathcal{V} = (\omega, v_o)$:
  - Our previous interpretation: translating at linear velocity v_o while rotating with ω along an axis passing through o
  - There are many equivalent interpretations depending on which reference point o we choose to represent the spatial velocity
  - If o is not on the actual screw axis, the above interpretation does not reflect the actual physical motion (just an equivalent way to represent rigid body velocity).
- Examples:

From Screw Motion to Twist

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$
- Fix a reference frame {A} with origin o_A.
- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^Av_{o_A})$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$$\omega = \hat{s}\dot{\theta} \qquad v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

From Twist to Screw Motion

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\dot{\theta}$
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)

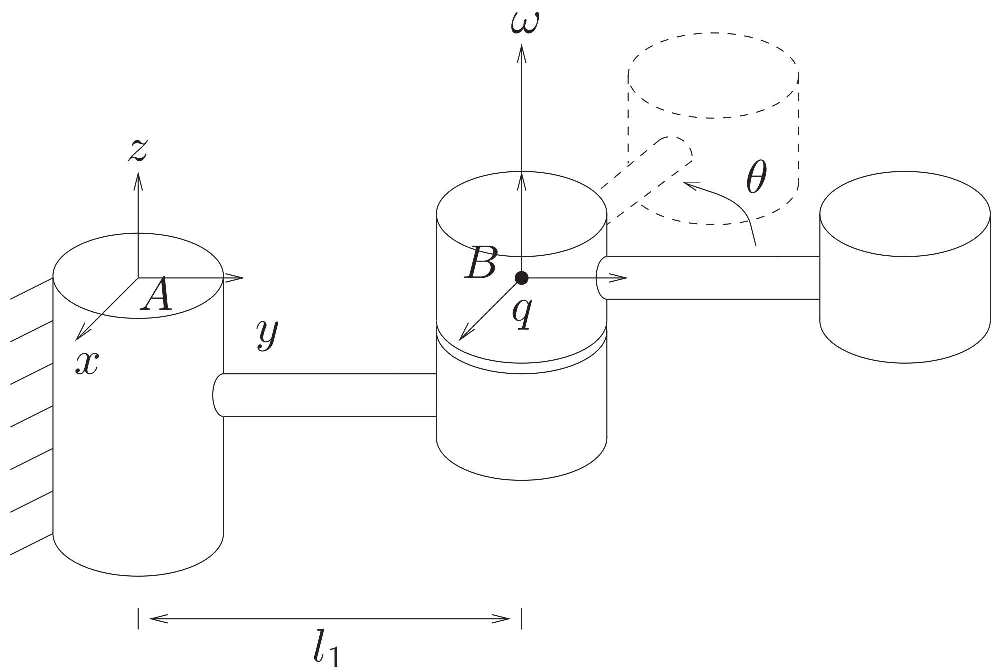
$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

  - If $\omega \neq 0$:

$$\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}$$

Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$?



- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

Screw Representation of a Twist

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

  - In this notation, we think of $\hat{\mathcal{S}}$ as the twist associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

More Discussions