MEE5114 Advanced Control for Robotics

Lecture 3: Operator View of Rigid-Body Transformation

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

- Rotation Operation
- Rigid-Body Transformation Operation

Outline

- Rotation Operation
- Rigid-Body Transformation Operation

Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.
- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n\times n}$ such that $\omega \times p = [\omega]p$

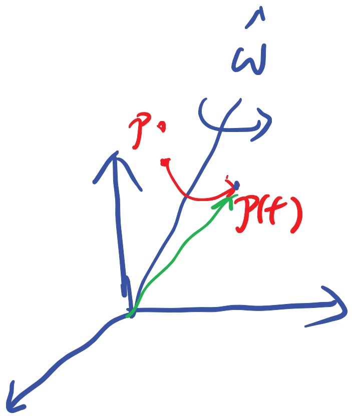
$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric
- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω
- The set of skew-symmetric matrices in: $so(n) \triangleq \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$
- We are interested in case $n = 2, 3$ $\quad R \in SO(3)$

Rotation Operation via Differential Equation

- Consider a point initially located at p_0 at time $t = 0$

- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by



$$\dot{p}(t) = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \tag{1}$$

linear velocity at time t

$\vec{op}$

Recall: $\dot{x} = Ax$, $x(0) = p_0$
$\Rightarrow x(t) = e^{At} p_0$

let $p(t) = x(t)$, $A \leftarrow [\hat{\omega}]$

- This is a linear ODE with solution: $p(t) = e^{[\hat{\omega}]t} p_0$

- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta} p_0$

- $\text{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

Rotation Matrix as a Rotation Operator (1/2)

$SO(3)$ $so(3)$

- Every rotation matrix R can be written as $R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

- To apply the rotation operation, all the vectors/matrices have to be expressed in the **same reference frame** (this is clear from Eq (1))

Rotation Matrix as a Rotation Operator (2/2)

- For example, assume $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{Rot}(\hat{x}; \pi/2)$
- Consider a relation $q = Rp$:

  $q = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Leftarrow p$

  - **Change reference frame interpretation** :

    two frames

    - R : orientation of {B} relative to {A} $\rightarrow {}^A R_B$
    - Then: p and q are coordinates of the same point in {B}, {A}

    one point

  - **Rotation operator interpretation**:
    - p and q are the coordinates of two points in the same frame

    ${}^A q' \Leftarrow \mathrm{Rot}(\hat{x}; \frac{\pi}{2})\, {}^A p$

    $p = {}^B a$

    $q = {}^A a$

    $\Leftarrow$ two points, one refrence

- Consider the frame operation:
  - Change of reference frame: $R_B = RR_A$ $\cancel{\Leftarrow}$ bad notation
  - Have one "frame object", and two refrence frames

  eg: Frame object: {A}, orientation in {o}, ${}^o R_A$, ${}^B R_A \Rightarrow {}^o R_A = {}^o R_B\, {}^B R_A$

  - Rotating a frame: $R'_A = RR_A$
  - two frame objects {A}, {A'}
  - One refrence frame.

  $R_{A'} = R\, R_A$ (rotation operation) $\Leftarrow$ more specifically, ${}^o R_{A'} = R\, {}^o R_A$

Rotation Operation in Different Frames (1/2)

coordinate free notation

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.

- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^A\hat{\omega}$ and {B}-frame coordinate ${}^B\hat{\omega}$. We know

$$ {}^A\hat{\omega} = {}^B R_A \, {}^B\hat{\omega} $$

- Let ${}^B\mathrm{Rot}({}^B\hat{\omega}, \theta)$ and ${}^A\mathrm{Rot}({}^A\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

the same physical operation

Rotation Operation in Different Frames (2/2)

- We have the relation:

$${}^{B}R_{A}\,{}^{A}R_{B} = I \Rightarrow ({}^{B}R_{A})^{-1} = {}^{A}R_{B}$$

$$\boxed{{}^{A}\mathrm{Rot}({}^{A}\hat{\omega},\theta) = {}^{A}R_{B}\,{}^{B}\mathrm{Rot}({}^{B}\hat{\omega},\theta)\,{}^{B}R_{A}}$$

- 1°: approach: two points $p \xrightarrow{\mathrm{Rot}()} p'$ ⇒ choose {A} frame: $\boxed{{}^{A}p' = {}^{A}\mathrm{Rot}({}^{A}\hat{\omega},\theta)\,{}^{A}p}$

choose {B} frame: ${}^{B}p' = {}^{B}\mathrm{Rot}({}^{B}\hat{\omega},\theta)\,{}^{B}p$

$\Rightarrow {}^{A}R_{B}\,{}^{B}p' = {}^{A}R_{B}\,\mathrm{Rot}({}^{B}\hat{\omega},\theta)\,{}^{B}R_{A}\,{}^{A}p$

$\Rightarrow {}^{A}p' = \boxed{{}^{A}R_{B}\,\mathrm{Rot}({}^{B}\hat{\omega},\theta)\,{}^{B}R_{A}}\,{}^{A}p$

- 2°: Identity: for any $a \in \mathbb{R}^3$, and any $R \in SO(3)$

$Ra \in \mathbb{R}^3 \Rightarrow \boxed{[Ra] = R[a]R^T}$

verify yourself (plug in definition)

$$\mathrm{Rot}({}^{A}\hat{\omega};\theta) = e^{[{}^{A}\hat{\omega}]\theta} = e^{[{}^{A}R_{B}\,{}^{B}\hat{\omega}]\theta} = e^{{}^{A}R_{B}[{}^{B}\hat{\omega}]\,{}^{B}R_{A}\,\theta} = {}^{A}R_{B}\,e^{[{}^{B}\hat{\omega}]\theta}\,{}^{B}R_{A}$$

Q1: $e^{TAW} \cancel{\neq} Te^{A}W$

$$e^{TAT^{-1}} = I + TAT^{-1} + \frac{TAT^{-1}TAT^{-1}}{2!} + \cdots = Te^{A}T^{-1}$$

Outline

- Rotation Operation
- Rigid-Body Transformation Operation

Rigid Transformation via Differential Equation (1/3)

- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(1). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta)p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}]p(t)$ with $p(0) = p_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Special Euclidean Group

Rigid Transformation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion

$\mathcal{S} = (\hat{s}, h, q)$

- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed ($\dot{\theta} = 1$). Let the corresponding twist be $\mathcal{V} = \mathcal{S} = (\omega, v)$. The motion can be described by the following ODE.

$\dot{\tilde{p}}(t) = [\quad] \cdot \tilde{p}(t)$

$$\dot{p}(t) = \underbrace{\omega \times p(t)}^{[\omega]p(t)} + v \quad \Rightarrow \quad \underbrace{\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}}_{\dot{\tilde{p}}(t)} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \tag{2}$$

4×1, 3×1, 1×1

Homogeneous coordinate

$p(t) \longrightarrow \tilde{p}(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix}$

$\dot{p}(t) \longrightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}$

- Solution to (2) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\underbrace{\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}}^{\tilde{A} \in \mathbb{R}^{4\times 4}} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

$\tilde{p}(t) = e^{\tilde{A}t} \tilde{p}(0)$

4×4 $\in SE(3)$ → rigid body operator

Rigid Transformation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation $\leftarrow \in \mathbb{R}^{4\times 4}$
  $\in 6\times 1$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$ $\Leftrightarrow S = \{\hat{s}, h, q\}$

- With this notation, the solution to (2) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t

$se(3)$

$\nearrow so(3) = \{[\omega]\}$

$\forall [\omega] \in so(3) \rightarrow e^{[\omega]\theta} = R \in SO(3)$

- Similar to $so(3)$, we can define $se(3)$: $\forall [\mathcal{V}] \in se(3) \rightarrow e^{[\mathcal{V}]\theta} = T \in SE(3)$

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

$$\Downarrow$$

$$\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Rigid Trans. Operation in Frames

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree ,

  if nonunit velocity, $\mathcal{V} = S \cdot \dot{\theta}$

  For $T \in SE(3)$

  $T = e^{[S]\theta}$

- $\tilde{p}' = T\tilde{p}$: "rotate" p along screw axis $\mathcal{S}$ by θ degree

  more specially, ${}^{o}\tilde{p}' = {}^{o}T \cdot {}^{o}\tilde{p}$

- TT_A: "rotate" {A}-frame along $\mathcal{S}$ by θ degree $\Rightarrow$ {A'}

  ${}^{o}T \cdot {}^{o}T_A \rightarrow {}^{o}T_{A'}$

- Expression of T in another frame (other than {O}):

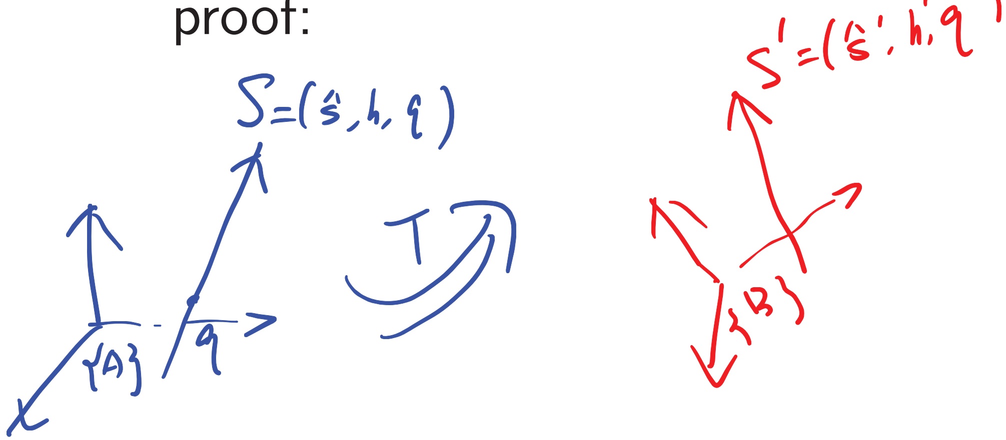
$$\underset{\text{operation in \{O\}}}{T} \quad \leftrightarrow \quad \underset{\text{operation in \{B\}}}{T_B^{-1}TT_B}$$

Rigid Operation for Screw Axis

- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\boxed{\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}}$$

proof:



Note: $T = e^{[\hat{S}]\theta}$

Let's work with an arbitrary frame {A}

Let's assume a frame {B} is obtained by applying T to {A}

the coordinate of S in {A}, is the same as the coordinate of S' in {B}

i.e.: ${}^{A}S = {}^{B}S'$ ... ①

$\Rightarrow$ We also know ${}^{A}T_{B} = T \cdot {}^{A}T_{A}$

(T takes {A} to {B})

By ①

$${}^{A}X_{B}\,{}^{A}S = {}^{A}X_{B}\,{}^{B}S' = {}^{A}S'$$

$$\Rightarrow {}^{A}S' = {}^{A}X_{B}\,{}^{A}S$$

More Space

${}^{A}X_{B} = \left[Ad_{A_{T_{BS}}}\right] = \left[Ad_{T}\right] \in 6\times 6$