MEE5114 Advanced Control for Robotics

Lecture 4: Exponential Coordinate of Rigid Body Configuration

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

- Exponential Coordinate of $SO(3)$ rotation matrix
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3) homo transformation matrix
- Instantaneous Velocity of Moving Frames

Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)
- Instantaneous Velocity of Moving Frames

Towards Exponential Coordinate of $SO(3)$

- Recall the polar coordinate system of the complex plane: $\mathbb{C}$
  - Every complex number $z = x + jy = \rho e^{j\phi}$
  - Cartesian coordinate $(x, y) \leftrightarrow$ polar coorindate (ρ, ϕ)
  - For some applications, the polar coordinate is preferred due to its geometric meaning.

$\Rightarrow \{R^T R = I,\ \det(R) = 1\}$

- For any rotation matrix $R \in SO(3)$, it turned out $R = e^{[\hat{\omega}]\theta}$
  - $\hat{\omega}$: unit vector representing the axis of rotation
  - θ: the degree of rotation
  - $\hat{\omega}\theta$ is called the **exponential coordinate** for $SO(3)$.

Exponential Coordinate of $SO(3)$

Proposition 1 (Exponential Coordinate $\leftrightarrow$ SO(3)).

- For any unit vector $[\hat{\omega}] \in so(3)$ and any $\theta \in \mathbb{R}$,

valid rotation matrix

$$e^{[\hat{\omega}]\theta} \in SO(3)$$

- For any $R \in SO(3)$, there exists $\hat{\omega} \in \mathbb{R}^3$ with $\|\hat{\omega}\| = 1$ and $\theta \in \mathbb{R}$ such that

$$R = e^{[\hat{\omega}]\theta}$$

$$\begin{aligned} &\text{exp:} \quad [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3) \\ &\text{log:} \quad R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3) \end{aligned}$$

- The vector $\hat{\omega}\theta$ is called the *exponential coordinate* for R
- The exponential coordinates are also called the canonical coordinates of the rotation group $SO(3)$

Rotation Matrix as Forward Exponential Map

- Exponential Map: By definition

$$e^{[\omega]\theta} = I + \theta[\omega] + \frac{\theta^2}{2!}[\omega]^2 + \frac{\theta^3}{3!}[\omega]^3 + \cdots$$

- **Rodrigues' Formula**: Given any unit vector $[\hat{\omega}] \in so(3)$, we have

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

Fact: if $\|\hat{w}\| = 1$, then $[\hat{w}] = -[\hat{w}]^T$, $[\hat{w}]^3 = -[\hat{w}]$, $[\hat{w}]^4 = [\hat{w}]^3[w] = -[w]^2$

$$e^{[\hat{w}]\theta} = I + [\hat{w}]\theta + \frac{\theta^2}{2}[\hat{w}]^2 + \frac{\theta^3}{3!}(-[\hat{w}]) + \frac{\theta^4}{4!}(-[\hat{w}]^2) + \cdots$$

$$= I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)}_{\sin\theta}[\hat{w}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta}[\hat{w}]^2$$

Examples of Forward Exponential Map

- Rotation matrix $\underset{\sim}{R_x(\theta)}$ (corresponding to $\hat{x}\theta$)

$$Rot(\hat{x},\theta) = e^{[\hat{x}]\theta} = I + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [\hat{x}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation matrix corresponding to $\underset{\sim}{(1,0,1)^T} \leftarrow$ exp coordinate

$$\hat{\omega} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}, \quad \theta = \sqrt{2}$$

Logarithm of Rotations

- If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and set $\hat{\omega}$ equal to one of the following

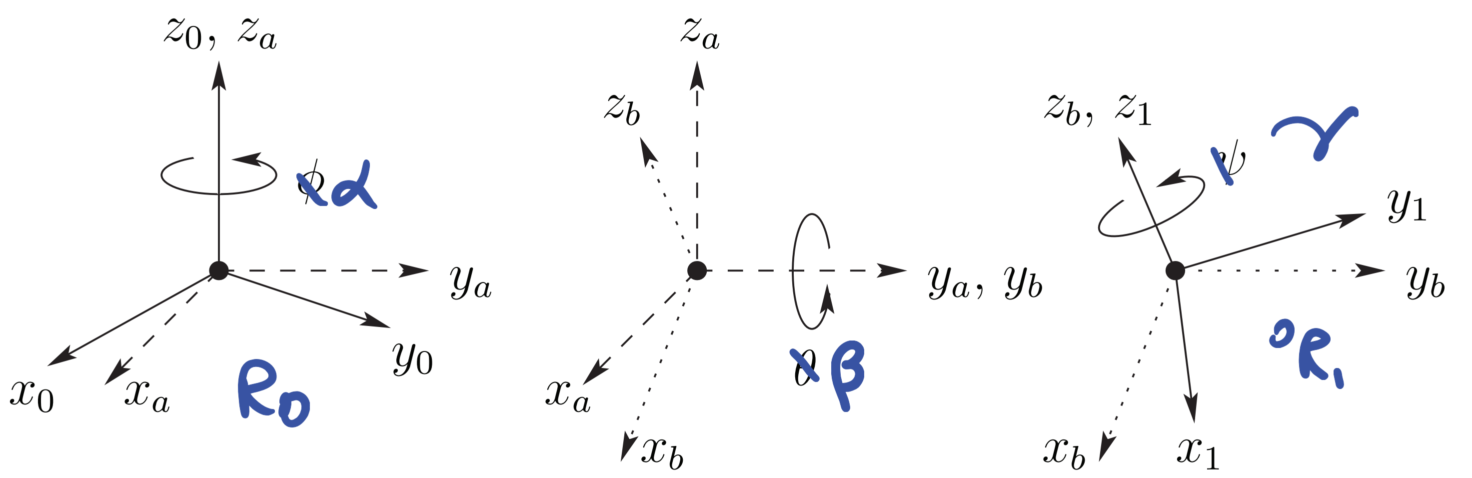
$$\frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

- Otherwise, $\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(R) - 1)\right) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)$

Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)
- Instantaneous Velocity of Moving Frames

Euler Angle Representation of Rotation



- A common method of specifying a rotation matrix is through three independent quantities called **Euler Angles**.
- Euler angle representation
  - Initially, frame $\{0\}$ coincides with frame $\{1\}$
  - Rotate $\{1\}$ about $\hat{\mathrm{z}}_0$ by an angle α, then rotate about $\hat{\mathrm{y}}_{\mathrm{a}}$ axis by β, and then rotate about the $\hat{\mathrm{z}}_{\mathrm{b}}$ axis by γ. This yields a net orientation ${}^0R_1(\alpha,\beta,\gamma)$ parameterized by the ZYZ angles (α,β,γ)
  - ${}^0R_1(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)$ $\Leftarrow$ ${}^0R_1(\alpha,\beta,\gamma) = e^{[\hat{\omega}]\theta}$

$\hat{\omega}\theta = \log({}^0R_1)$

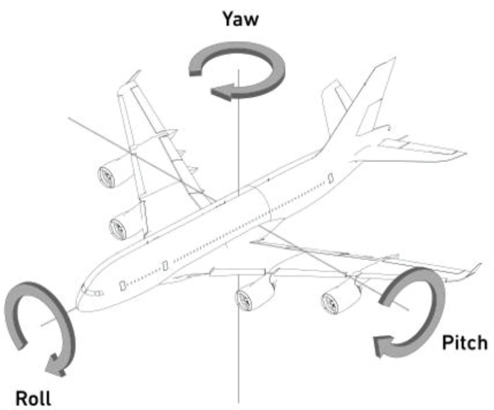
Other Euler-Like Parameterizations

- Other types of Euler angle parameterization can be devised using different ordered sets of rotation axes

- Common choices include:
  - ZYX Euler angles: also called *Fick angles* or yaw, pitch and roll angles

    $R_z(\alpha) R_y(\beta) R_x(\gamma)$

  - YZX Euler angles (Helmholtz angles)



Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)
- Instantaneous Velocity of Moving Frames

Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1.

For any $\mathcal{V} = (\omega, v)$ *and* $\theta \in \mathbb{R}$*, we have* $e^{[\mathcal{V}]\theta} \in SE(3)$

- *Case 1* $(\omega = 0)$: $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- *Case 2* $(\omega \neq 0)$: *without loss of generality assume* $\|\omega\| = 1$. *Then*

$$e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{ with } G(\theta) = I\theta + (1 - \cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2 \quad (1)$$

$$\mathcal{V} \in \mathbb{R}^{6\times 1} = \begin{bmatrix} \omega \\ v \end{bmatrix}, \quad [\mathcal{V}] = \begin{array}{cc} 3\times 3 & 3\times 1 \\ \left[\begin{array}{c|c} [\omega] & v \\ \hline 0 & 1 \end{array}\right] & \end{array} \in \mathbb{R}^{4\times 4}, \quad \underbrace{e^{[\mathcal{V}]\theta}} \in SE(3)$$

Forward map (exponential) from $\underbrace{se(3)}_{\substack{\text{twist} \\ \text{screw axis}}} \xrightarrow{\exp(\cdot)} \underbrace{SE(3)}_{\substack{\text{transformation} \\ \text{matrix}}}$

Log of $SE(3)$: from Rigid-Body Motion to Twist

Theorem 2. $T=\left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]$

Given any $T = (R, p) \in SE(3)$, one can always find twist $\mathcal{V} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{V}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm Algorithm:

- If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.
- Otherwise, use matrix logarithm on $SO(3)$ to determine ω and θ from R. Then v is calculated as $v = G^{-1}(\theta)p$, where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2}\right)[\omega]^2$$

Exponential Coordinates of Rigid Transformation

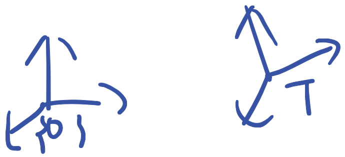
- To sum up, screw axis $\mathcal{S} = (\omega, v)$ can be expressed as a normalized twist; its matrix representation is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

- A point started at $p(0)$ at time zero, travel along screw axis $\mathcal{S}$ at unit speed for time t will end up at $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Given $\mathcal{S}$ we can use Theorem 1 to compute $e^{[\mathcal{S}]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $\mathcal{S} = (\omega, v)$ and θ such that $e^{[\mathcal{S}]\theta} = T$.



$T = e^{[S]\theta}$ means

- We call $\mathcal{S}\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$

- Spatial velocity / twist : $\mathcal{V} = \begin{bmatrix} \omega \\ v_o \end{bmatrix}$ → ref point "o" may not move at all
  - ω : angular velocity
  - v_o : velocity of the body-fixed point currently coincides with o
  - velocity of any body-fixed point p is $v_p = v_o + \omega \times (\vec{op})$

- Twist in frames : given frame {B}, {O} with relation ${}^{O}T_B = (R, p)$

  - $\underset{6\times 1}{{}^{O}\mathcal{V}} = \begin{bmatrix} {}^{O}\omega \\ {}^{O}v_{\underbrace{O}_{\text{origin of \{O\} frame}}} \end{bmatrix}$, ${}^{B}\mathcal{V} = \begin{bmatrix} {}^{B}\omega \\ {}^{B}v_{\underbrace{B}_{\text{origin of \{B\}}}} \end{bmatrix}$

$${}^{O}\mathcal{V} = \underbrace{{}^{O}X_B}_{6\times 6} \, {}^{B}\mathcal{V}$$

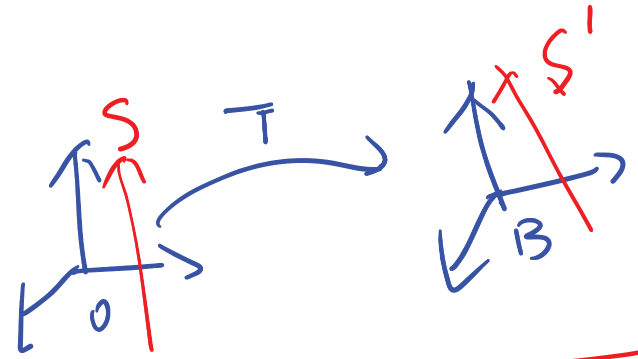
→ change of coordinate for twist

$${}^{O}X_B = [Ad_{{}^{O}T_B}] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

for any $T \in SE(3)$, $\Rightarrow [Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$

$\;\;\; \downarrow (R, p) \qquad\qquad\quad 6\times 6$

$\underset{\sim\sim\sim}{T = e^{[\hat{S}]\theta}} \longrightarrow \underset{\text{twist}}{S} \longrightarrow S'$



- **screw axis** : - $S = (\hat{s}, h, q) \longleftrightarrow \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ h\hat{s} - \hat{s} \times q \end{bmatrix}$

$\dot{\theta} = 1$

- all rigid body motion can be "thought of" as screw motion rotation & linear motion along the same axis

- We typically write $\mathcal{V} = S\dot{\theta}$

- Exponential coordinate & rotation operator (with {0})
  - ODE for rotation: $\dot{p} = w \times p = [w] p \Rightarrow p(t) = e^{[w]t} p(0)$
  - if $w = \hat{w}$, unit vector, $t = \theta$
  - $\hat{w}\theta \Leftrightarrow R = e^{[\hat{w}]\theta} \in SO(3)$. $\hat{w}\theta$ is exp. coordinate of R
  - $p' = \underbrace{e^{[\hat{w}]\theta}}_{R} p$
  - Given frame {A}, $R_A = [\hat{x}_A, \hat{y}_A, \hat{z}_A]$, then $\underbrace{R R_A}$ ~~means~~ rotate {A} about $\hat{w}$ by θ

    $R \to e^{[\hat{w}]\theta}$
  - Expression of rotation operator R in another frame {B}.

    R — rotation matrix in {0}

    $R_B^{-1} R R_B$ — rotation in {B}

- Exp coordinate for rigid body motion {with {O}}

- ODE : $\dot{p} = v + \underbrace{(\omega \times)}_{[w]} p$ under twist $\mathcal{V} = \begin{bmatrix} w \\ v \end{bmatrix}$

- $\Rightarrow \dot{\tilde{p}} = \underbrace{\begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix}}_{[\mathcal{V}] \in \mathbb{R}^{4\times4}} \tilde{p} \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t} \tilde{p}(0)$

- consider unit-twist

- $se(3) \underset{\log}{\overset{\exp}{\rightleftarrows}} SE(3)$

  S $\qquad T = e^{[S]\theta}$

- $\tilde{p}' = T \tilde{p}$: rotate p along S by θ degree

- $T \cdot T_A$: rotate frame {A} along . . . .

- rigid operation on screw axis : $\underbrace{[Ad_T]}_{6\times6} \cdot \underline{S_1}_{6\times1}$ means rotate screw axis S_1 along S by θ degree

  $T = e^{[S]\theta}$

Outline – expression of T in {B}

$$T \longleftrightarrow T_B^{-1} T T_B$$

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)
- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Rotating Frame

- {A} frame is rotating with orientation $R_A(t)$ and velocity $\omega_A(t)$ at time t (Note: everything is wrt {O}-frame)
- Let $\hat{\omega}\theta = \log(R_A(t))$ be its exp. coordinate.

  $\hat{\omega}(t)\theta$

  - Note: $\hat{\omega}\theta$ means $R_A(t)$ can be obtained from the reference frame (say {O}-frame) by rotating about $\hat{\omega}$ by θ degree.
  - $\hat{\omega}\theta$ only describes the current orientation of {A} relative to {O}, it does not contain info about how the frame is rotating at time t.
- What is the relation between $\omega_A(t)$ and $R_A(t)$?

$$\frac{d}{dt}R_A(t) = [\omega_A(t)]R_A(t) \Rightarrow [\omega_A(t)] = \dot{R}_A(t)R_A^{-1}(t)$$

$$R_A(t) = [x_A(t) \quad y_A(t) \quad z_A(t)]$$

$$\dot{x}_A(t) = w_A \times x_A(t) = [w_A]\, x_A(t), \quad \dot{y}_A(t) = [w_A]\, y_A(t), \quad \dot{z}_A(t) = [w_A]\, z_A$$

$$\Rightarrow \quad \dot{R}_A = [w_A]\, R_A \quad \Rightarrow [w_A] = \dot{R}_A(t) \cdot R_A(t)^{-1}$$

Instantaneous Velocity of Moving Frame

- {A} moving frame with configuration $T_A(t)$ at time t undergoes a rigid body motion with velocity $\mathcal{V}_A(t) = (\omega, v)$ (Note: everything is wrt {O}-frame)
- The exponential coordinate $\hat{\mathcal{S}}\theta = \log(T_A(t))$ only indicates the current configuration of {A}, and does not tell us about how the frame is moving at time t.
- What is the relation between $\mathcal{V}_A(t)$ and $T_A(t)$?

$$\frac{d}{dt}T_A(t) = [\mathcal{V}_A(t)]T_A(t) \Rightarrow [\mathcal{V}_A(t)] = \dot{T}_A(t)T_A^{-1}(t)$$

$T_A = \begin{bmatrix} R_A & p_A \\ 0 & 1 \end{bmatrix}$, a frame can be determined by direction vector of axes, and origin p

free vector   point

In homogeneous coordinate: $\tilde{x}_A = \begin{bmatrix} x_A \\ 0 \end{bmatrix}$, $\tilde{y}_A = \begin{bmatrix} y_A \\ 0 \end{bmatrix}$; $\tilde{z}_A = \begin{bmatrix} z_A \\ 0 \end{bmatrix}$

$\tilde{p}_A = \begin{bmatrix} p_A \\ 1 \end{bmatrix}$

$\dot{\tilde{x}}_A = [\mathcal{V}_A]\tilde{x}_A$, $\dot{\tilde{y}}_A = [\mathcal{V}_A]\tilde{y}_A \cdots \Rightarrow \dot{T}_A = [\mathcal{V}_A]T_A \Longrightarrow [\mathcal{V}_A] = \dot{T}_A T_A^{-1}$

More Space

More Space