MEE5114 Advanced Control for Robotics

Lecture 5: Kinematics of Open Chain

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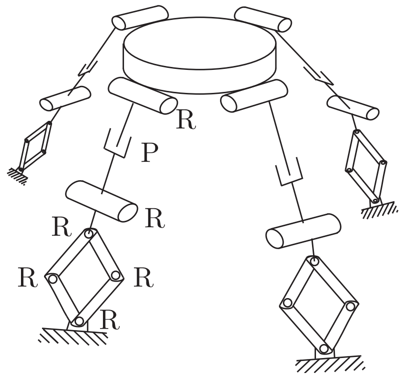
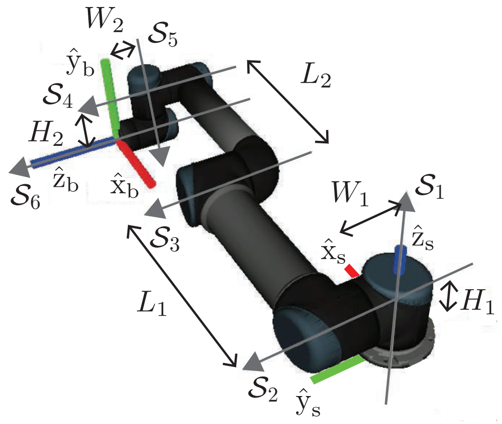
Outline

- Background
- Forward Kinematics: Product of Exponential Formula
- Jacobian
- Geometric Jacobian Derivation
- Analytic Jacobian

Kinematics

Kinematics is a branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the mass of each or the forces that caused the motion





- **Forward Kinematics**: calculation of the configuration $T = (R, p)$ of the end-effector frame from joint variables $\theta = (\theta_1, \ldots, \theta_n)$

- **Velocity Kinematics**: Deriving the Jacobian matrix: linearized map from the joint velocities to the spatial velocity of the end-effector

Basic Setup (1/3)

- Suppose that the robot has n joints and n links. Each joint has one degree of freedom represented by joint variable θ_i, $i = 1, \ldots, n$
  - θ_i: the joint angle (Revolute joint) or joint displacement (Primatic joint)

- Specify a fixed frame {s}: also referred to as frame {0}

- Attach frame $\{i\}$ to link i at joint i, for $i = 1, \ldots, n$

- Attach frame {b} at the end-effector: sometimes referred to as frame {n+1}

- ${}^{i}\mathcal{S}_i$: screw axis of joint i expressed in frame {i}

- ${}^{0}\mathcal{S}_i$: screw axis of joint i expressed in fixed frame {0} (i.e. frame {s})

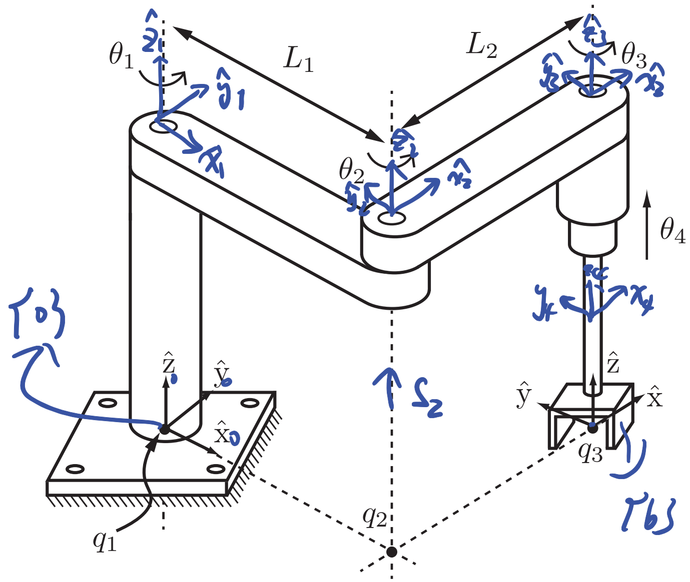
Basic Setup (2/3)

- Illustration Example:

e.g.: ${}^0S_1 = (\omega_1, v_1)$, ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$v_1 = -\omega_1 \times q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow {}^0S_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$



${}^2S_2 = ({}^2\omega_2, {}^2v_2)$, ${}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^2v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0S_2 = {}^0X_2 \cdot {}^2S_2$

$\rightarrow$ depends on θ_1, θ_2

Goal: Find expressions of ${}^0T_b$ in terms of $\theta_1, \theta_2, \theta_3, \theta_4$

Basic Setup (3/3)

- For simplicity, we write configuration as T_{sb}, which is the same as ${}^{s}T_{b}$. Similarly, $T_{ij} = {}^{i}T_{j}$

- Note: ${}^{i}\mathcal{S}_i$ does not change when the robot moves (i.e. when θ changes), but ${}^{0}\mathcal{S}_i$ depends on $\theta_1, \ldots, \theta_i$. Sometimes, we write out the dependency explicitly, i.e. ${}^{0}\mathcal{S}_i(\theta_1, \ldots, \theta_i)$

- Define home position: $\theta_1 = 0, \ldots, \theta_n = 0$. This is the configuration when all the joint angles are zero. One can also choose other *fixed* angles as the home position

- Define ${}^{0}\bar{\mathcal{S}}_i = {}^{0}\mathcal{S}_i(0, \ldots, 0)$: the screw axis of joint i expressed in frame $\{0\}$, when the robot is at the home position.

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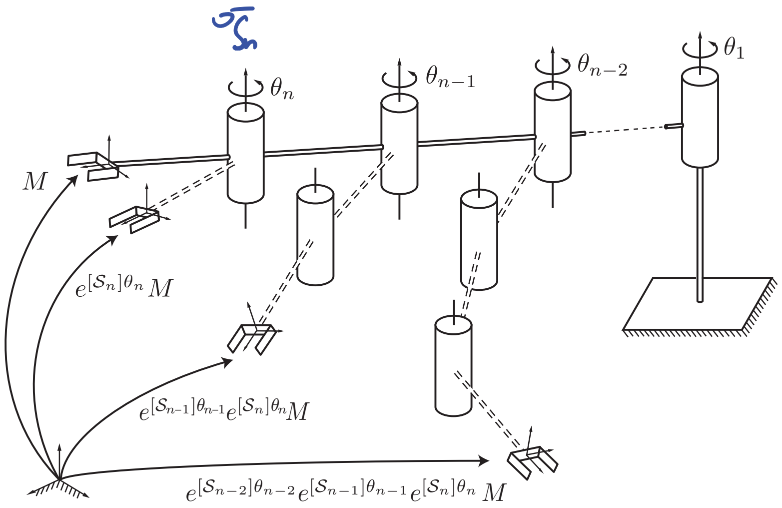
Product of Exponential: Main Idea



- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$
- Compute $M \triangleq T_{sb}(0, \ldots, 0)$: the configuration of end-effector when the robot is at home position
- Apply screw motion to joint n: $T_{sb}(0, \ldots, 0, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$ ← new configuration
- Apply screw motion to joint $n-1$ to obtain:

$$T_{sb}(0, \ldots, 0, \theta_{n-1}, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_{n-1}]\theta_{n-1}} e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

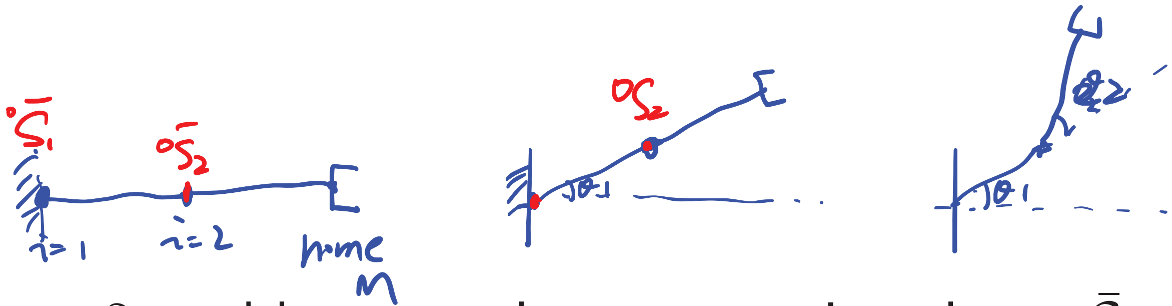
- After n screw motions, the overall forward kinematics:

$$T_{sb}(\theta_1, \ldots, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} \cdots e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

product of exp.

PoE: Screw Motions in Different Order (1/2)

- PoE was obtained by applying screw motions along screw axes ${}^0\bar{\mathcal{S}}_n$, ${}^0\bar{\mathcal{S}}_{n-1}$, ... What happens if the order is changed?



- For simplicity, assume that $n = 2$, and let us apply screw motion along ${}^0\bar{\mathcal{S}}_1$ first:
  - $T_{sb}(\theta_1, 0) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M$
  - Now screw axis for joint 2 has been changed. The new axis ${}^0\mathcal{S}_2 = {}^0\mathcal{S}_2(\theta_1, 0) \neq {}^0\bar{\mathcal{S}}_2$.

$${}^0\bar{S}_2 \xrightarrow{T=e^{[\bar{S}_1]\theta_1}} \underbrace{{}^0S_2(\theta_1)}_{6\times 1} = \underbrace{[\mathrm{Ad}_T]}_{6\times 6}\underbrace{{}^0\bar{S}_2}_{6\times 1}$$, where $T = e^{[{}^0\bar{S}_1]\theta_1}$

① Fact: If $S' = [\mathrm{Ad}_T]\, S \iff \underbrace{[S']}_{4\times 4} = \underbrace{T}_{4\times 4}\underbrace{[S]}_{4\times 4}\underbrace{T^{-1}}_{4\times 4}$

$[R\omega] = R[\omega]R^{-1}$

PoE: Screw Motions in Different Order (2/2)

- $T_{sb}(\theta_1, \theta_2) = e^{[{}^0\mathcal{S}_2]\theta_2} T_{sb}(\theta_1, 0)$

From the "Fact": $e^{[{}^0\bar{S}_2]\theta_2} = e^{T[{}^0\bar{S}_2]T^{-1}\theta_2} = T e^{[{}^0\bar{S}_2]\theta_2} T^{-1}$

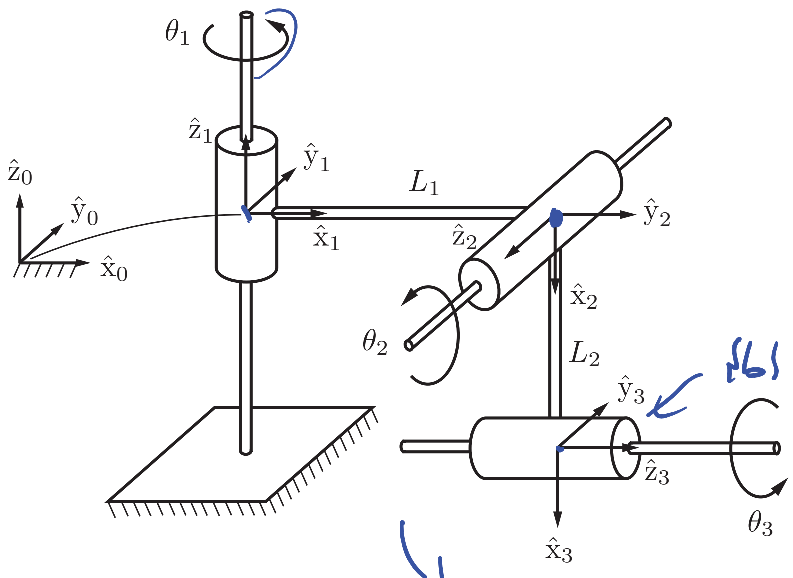
where $T = e^{[{}^0\hat{S}_1]\theta_1}$

$$T_{sb}(\theta_1, \theta_2) = T e^{[{}^0\bar{S}_2]\theta_2} \underline{T^{-1} e^{[{}^0\bar{S}_1]\theta_1}} M$$

$$= e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} M$$

PoE Example: 3R Spatial Open Chain

Find ${}^sT_b = {}^0T_3 = T_{sb}(\theta_1, \theta_2, \theta_3)$



assume this is home position

step 1: $M = \left[\begin{array}{ccc|c} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$

Step 2: ${}^0\bar{S}_1 = (\omega_1, v_1)$. $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, pick $q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0\bar{S}_2 = (\omega_2, v_2)$, $\omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, pick $q_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = -\omega_2 \times q_2 = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$

$\Rightarrow {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ L_1 \end{bmatrix}$

${}^0\bar{S}_3 = \ldots$

$T_{sb}(\theta_1, \theta_2, \theta_3) = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_3]\theta_3} M$

Outline

Jacobian

- Given a multivariable function $x = f(\theta)$, where $x \in \mathbb{R}^m$ and $\theta \in \mathbb{R}^n$. Its **Jacobian** is defined as

$$J(\theta) \triangleq \left[\frac{\partial f}{\partial \theta}(\theta)\right] \triangleq \left[\frac{\partial f_i}{\partial \theta_j}\right]_{i\le m, j\le n} \in \mathbb{R}^{m\times n}$$

m×n matrix

eg: $x = 2\theta_1^2 + 3\theta_2^3$

Jacobian

$$\dot{x} = \frac{\partial x}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial x}{\partial \theta_2}\dot{\theta}_2 = \begin{bmatrix}\frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$$

- If x and θ are both a function of time, then their velocities are related by the Jacobian:

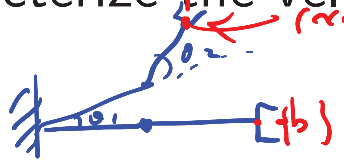
$$\dot{x} = \left[\frac{\partial f}{\partial \theta}(\theta)\right]\frac{d\theta}{dt} = J(\theta)\dot{\theta} = \begin{bmatrix}\underbrace{\frac{\partial f}{\partial \theta_1}}_{J_1(\theta)} & \underbrace{\frac{\partial f}{\partial \theta_2}}_{J_2(\theta)} & \cdots & \underbrace{\frac{\partial f}{\partial \theta_n}}_{J_n(\theta)}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n\end{bmatrix}$$

- Let $J_i(\theta)$ be the ith column of J, then $\dot{x} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$
  - $J_i(\theta)$ is the velocity of x due to $\dot{\theta}_i$ (while $\dot{\theta}_j = 0$ for all $j \neq i$)

let $\dot{\theta}_i = 1$, $\dot{\theta}_1 = \cdots \dot{\theta}_{i-1} = \dot{\theta}_{i+1} = \cdots = 0 \Rightarrow \boxed{\dot{x} = J_i(\theta)}$

Analytic vs. Geometric Jacobian

- A straightforward way to characterize the velocity kinematics is through the **Analytical Jacobian**



- Express the configuration of $\{b\}$ using a minimum set of coordinates x. For example:
  - (x_1, x_2, x_3): Cartesian coordinates or spherical coordinate of the origin of the $\{b\}$
  - (x_4, x_5, x_6): Euler angles or exponential coordinate of the orientation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = f(\theta_1, \theta_2) \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

- Write down the forward kinematics using the minimum set of coordinate x: $x = f(\theta)$

depends on coordinate sys
(Cartesian, spherical ...)

- Analytical Jacobian can then be computed as $J_a(\theta) = \left[\frac{\partial f}{\partial \theta}(\theta)\right]$

- The analytical Jacobian J_a depends on the local coordinates system of $SE(3)$

Analytic vs. Geometric Jacobian

- **Geometric Jacobian** directly finds relation between joint velocities and end-effector twist:

angular velocity

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}, \quad \text{where } J(\theta) \in \mathbb{R}^{6\times n}$$

- Note: $\mathcal{V} = (\omega, v)$ is NOT a derivative of any position variable, i.e. $\mathcal{V} \neq \frac{dx}{dt}$ (regardless what representation x is used) because the angular velocity is not the derivative of any time varying quantity.

recall PoE formula: $T_{ef} = T(\theta_1, \theta_2) = (e^{[S_1]\theta_1} e^{\cdots} M)$

- Analytical Jacobian J_a destroys the natural geometric structure of the rigid body motion;

- Geometric Jacobian can be used to derive analytical Jacobian.

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Derivation of Geometric Jacobian (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$$

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

depend on θ because
screw axis depends on θ

Derivation of Geometric Jacobian (2/3)

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.
- The simplest way to write Jacobian is to use local coordinate:

$$^{i}J_i = {}^{i}S_i, \quad i = 1, \dots, n$$

- In fixed frame $\{0\}$, we have

$$^{0}J_i(\theta) = {}^{0}X_i(\theta)\, {}^{i}S_i, \quad i = 1, \dots, n \tag{1}$$

- Recall: $^{0}X_i$ is the change of coordinate matrix for spatial velocities.
- Assume $\theta = (\theta_1, \dots, \theta_n)$, then

$$^{0}T_i(\theta) = e^{[{}^{0}\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^{0}\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^{0}X_i(\theta) = \left[\mathrm{Ad}_{{}^{0}T_i(\theta)}\right] \tag{2}$$

Derivation of Geometric Jacobian (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i=1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i=2$, ${}^0S_2(\theta) = {}^0S_2(\theta_1) = \left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$

    operator: rotate about ${}^0\bar{S}_1$ by θ_1 degree

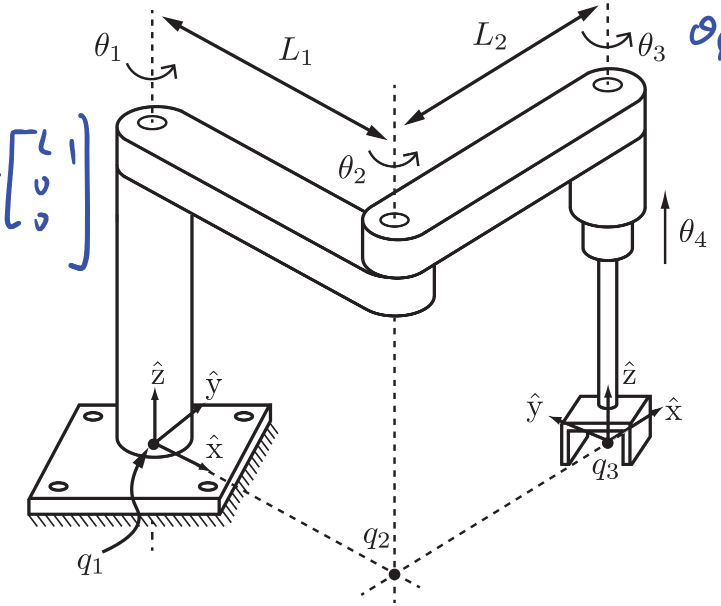
  - For general i, we have

$$
\begin{aligned}
{}^0J_i(\theta) = {}^0S_i(\theta) &= \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i \\
\text{where} \quad \hat{T}(\theta_1,\ldots,\theta_{i-1}) &\triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}}
\end{aligned}
\tag{3}
$$

Geometric Jacobian Example

Assume current pos is $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, $\theta_3 = 0$, $\theta_4 = 0$

1°: Find screw axis at home position in $\{0\}$ frame



$${}^0\bar{S}_1 = (\omega_1, v_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\bar{S}_2 = (\omega_2, v_2), \quad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$${}^0\bar{S}_3 = (\omega_3, v_3), \quad {}^0\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_3 = \begin{bmatrix} L_1 + L_2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 - L_2 \\ 0 \end{bmatrix}$$

$${}^0\bar{S}_4 = (\omega_4, v_4), \quad \omega_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

1st column 2nd column 3rd column

$$T_3 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} e^{[{}^0\bar{S}_3]\theta_3}$$

$$\Rightarrow {}^0J(\theta) = \Big[\underbrace{{}^0\bar{S}_1}_{1\times 6} \;\vdots\; \underbrace{[Ad_{T_1}]}_{T_1 = e^{[{}^0\bar{S}_1]\theta_1}} {}^0\bar{S}_2 \;\vdots\; \underbrace{[Ad_{T_2}]}_{T_2 = e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}} {}^0\bar{S}_3 \;\vdots\; [Ad_{T_3}]\, {}^0\bar{S}_4 \Big] \in 6\times 4$$

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$\mathcal{V} = {}^0J(\theta)\,\dot{\theta}$

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Analytic Jacobian

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be Cartesian + Euler angle of end-effector frame
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame
- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

  - $E(x)$ can be easily found with given parameterization x

Analytic Jacobian Example I (1/3)

- Suppose $x = \begin{bmatrix} x_R \\ x_p \end{bmatrix}$, where $x_R = (\alpha, \beta, \gamma)$ is the Z-Y-X Euler angles of the end-effector frame (relative to the inertia frame), and $x_p \in \mathbb{R}^3$ is the position of the origin of the end-effector frame (expressed in the inertia frame)

- Consider twist in world (inertia) frame: ${}^{O}\mathcal{V} = \begin{bmatrix} \omega \\ v_o \end{bmatrix} = J(\theta)\dot{\theta}$.

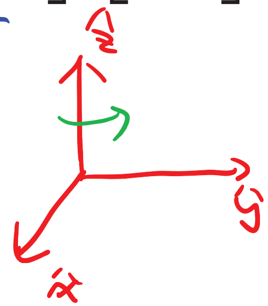
- We want to find analytic Jacobian, such that $\dot{x} = J_a(\theta)\dot{\theta}$

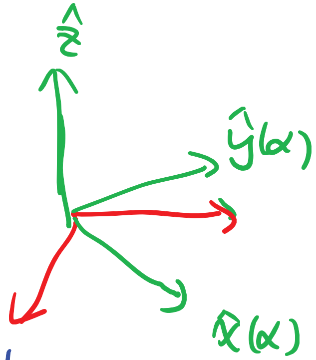
- Note: $\underbrace{{}^{O}R_b}_{\text{body orientation}} = R_z(\alpha)R_y(\beta)R_x(\gamma)$ $\Rightarrow$ ${}^{o}\dot{R}_b = \underbrace{\omega\times}\, {}^{o}R_b$

Analytic Jacobian Example I (2/3)

- One can show that: $\omega = \underbrace{\begin{bmatrix} 0 & -s_\alpha & c_\beta c_\alpha \\ 0 & c_\alpha & c_\beta s_\alpha \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{bmatrix} = B(x_R)\dot{x}_R$

Think about this as a 3R-robot





$$\omega = J_1\dot\alpha_1 + J_2\dot\beta + J_3\dot\gamma$$

$$= \underbrace{\begin{bmatrix} 0 & -S_\alpha & C_\beta C_\alpha \\ 0 & C_\alpha & C_\beta S_\alpha \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} \dot\alpha_1 \\ \dot\beta \\ \dot\gamma \end{bmatrix}$$

$$= B(x_R)\dot{x}_R$$

3R-robot with ${}^0\bar S_1 = \hat z$, ${}^0\bar S_2 = \hat y$, ${}^0\bar S_3 = \hat x$

$J_1 = {}^0\bar S_1 = \hat z$

$J_2 = \underbrace{[Ad_{T_1}]}\, {}^0\bar S_2 = [Ad_{T_1}]\,\hat y$

$T_1 = e^{[\hat z]\alpha}$

$J_3 = [Ad_{T_2}]\, {}^0\bar S_3 = [Ad_{T_2}]\,\hat x$

$T_2 = e^{[\hat z]\alpha} e^{[\hat y]\beta}$

Analytic Jacobian Example I (3/3)

- Be definition, we have

$$\dot{x}_R = B(x_R)^{-1}\omega, \quad \dot{x}_p = v_o + \omega \times x_p$$

$$\dot{x} = \begin{bmatrix} \dot{x}_R \\ x_p \end{bmatrix}$$

geometric Jacobian

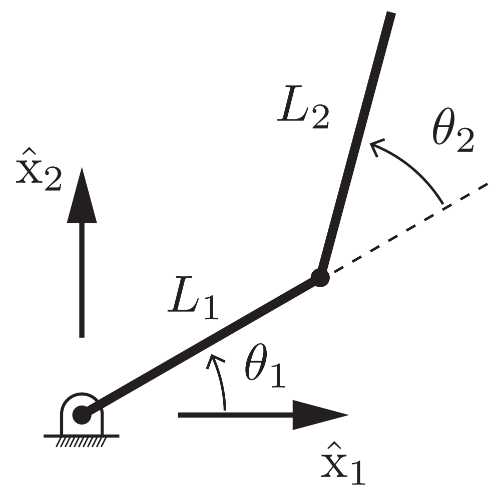
- $\dot{x} = \begin{bmatrix} B(x_R)^{-1} & 0 \\ -[x_p] & I \end{bmatrix} {}^{o}\mathcal{V} = E(x)J(\theta)\dot{\theta}$

$$\begin{bmatrix} \omega \\ v_o \end{bmatrix}$$

- If we use twist ${}^{b}\mathcal{V}$ in end-effector frame, the formula will be different.

Analytic Jacobian Example II

- $\begin{cases} x_1 = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ x_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \end{cases}$



More Discussions

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