MEE5114 Advanced Control for Robotics

Lecture 6: Rigid Body Dynamics

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Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Acceleration

- Given a rigid body with spatial velocity $\mathcal{V} = (\omega, v_o)$, its spatial acceleration is

$$\mathcal{A} = \dot{\mathcal{V}} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix}$$

  - Recall that: v_o is the velocity of the body-fixed particle coincident with frame origin o at the current time t.
  - Note: $\dot{\omega}$ is the angular acceleration of the body
  - $\dot{v}_o$ is not the acceleration of any body-fixed point!
  - In fact, $\dot{v}_o$ gives the rate of change in stream velocity of body-fixed particles passing through o

- Spatial Acceleration is a true vector (just like twist)

$$\mathcal{V}_{c/a} = \mathcal{V}_{b/a} + \mathcal{V}_{c/b} \quad \Rightarrow \quad \mathcal{A}_{c/a} = \mathcal{A}_{b/a} + \mathcal{A}_{c/b}$$

Spatial vs. Conventional Accel. (1/2)

Suppose: spatial velocity of rigid body given by $\begin{bmatrix} \omega \\ v_o \end{bmatrix}$, expressed at o at time t

- Let $r(t)$ be the position of a body-fixed particle that was at ~~origin~~ o ~~when $t=0$~~ i.e., at time t, $r(t)=0$, at $t+\Delta t$ $r(t) \neq 0$

- Note that: $\dot{r}(t) = v_o(t)$, but $\ddot{r}(t) \neq \dot{v}_o(t)$, where $\ddot{r}(t)$ is the conventional acceleration of the body-fixed point r

$\dot{p}(t) = V_o + \omega \times p(t)$

- In general, we have

$$\ddot{r}(t) = \dot{v}_o(t) + \omega(t) \times \dot{r}(t)$$

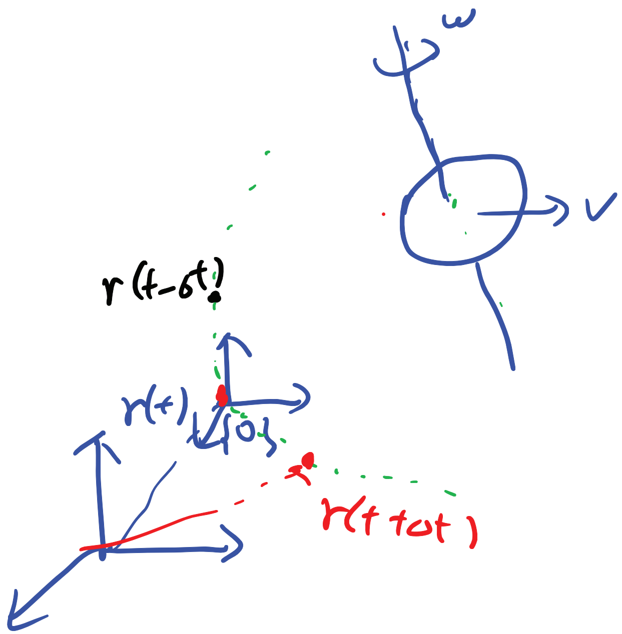
① $r(t)$ is body-fixed so at any time s

$\dot{r}(s) = V_o(s) + \omega(s) \times r(s)$

$\Rightarrow$ ①-ⓐ: let $s=t$, $\dot{r}(t) = V_o(t) + \omega(t) \times r(t)$
$= V_o(t)$

$\Rightarrow$ ①-ⓑ: let $s = t+\Delta t$

$\dot{r}(t+\Delta t) = V_o(t+\Delta t) + \omega(t+\Delta t) \times r(t+\Delta t)$



Spatial vs. Conventional Accel. (2/2)

② : For small Δt $\quad r(t+\Delta t) = \underline{r(t) + \dot{r}(t)\Delta t}$

$$\Rightarrow \underline{\dot{r}(t+\Delta t)} = v_o(t+\Delta t) + \omega(t+\Delta t) \times \dot{r}(t)\Delta t$$

③ $\dot{v}_o(t) = \lim_{\Delta t \to 0} \frac{v_o(t+\Delta t) - v_o(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\dot{r}(t+\Delta t) - \omega(t+\Delta t) \times \dot{r}(t)\Delta t - \dot{r}(t)}{\Delta t}$

$$= \ddot{r}(t) - \omega(t) \times \dot{r}(t)$$

Question: Given arbitrary reference frame $\{A\}$, arbitrary body-fixed point p, What's the relation between ${}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}$, ${}^A\mathcal{A} = \begin{bmatrix} {}^A\dot{\omega} \\ {}^A\dot{v}_A \end{bmatrix}$, and ${}^A\dot{p}$, ${}^A\ddot{p}$?

At time t, place frame $\{B\}$ with origin at $p(t)$, then we have

- $v_B = \dot{p}$ and $\dot{v}_B = \ddot{p} - \omega \times \dot{p}$ (by derivation above)

$$\Rightarrow {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix}, \quad {}^B\mathcal{A} = \begin{bmatrix} {}^B\dot{\omega} \\ {}^B\ddot{p} - {}^B\omega \times {}^B\dot{p} \end{bmatrix} \Rightarrow \begin{array}{l} {}^A\mathcal{A} = {}^A X_B \, {}^B\mathcal{A} \\ {}^A\tilde{p} = {}^A T_B \, {}^B\tilde{p} \leftarrow \text{homogeneous transformation} \end{array}$$

Spatial Acceleration in Plücker Coordinate Systems

- Fix an inertia frame {O}, let {B} be another frame (possibly moving)
- Consider a body with velocity $\mathcal{V}$ (wrt inertia frame), and ${}^O\mathcal{V}$ and ${}^B\mathcal{V}$ be its Plücker coordinates wrt {O} and {B}:

spacial velocity

$${}^O\!\left(\frac{d}{dt}\mathcal{V}\right) \qquad {}^B\!\left(\frac{d}{dt}\mathcal{V}\right) \qquad \frac{d}{dt}\left({}^O\mathcal{V}\right) \quad \frac{d}{dt}\left({}^B\mathcal{V}\right)$$

$${}^O\!A = {}^O\!\left(\frac{d\mathcal{V}}{dt}\right) \triangleq \lim_{\Delta t \to 0} \frac{{}^O\mathcal{V}(t+\Delta t) - {}^O\mathcal{V}(t)}{\Delta t} = \frac{d}{dt}\left({}^O\mathcal{V}\right)$$

$$\frac{d}{dt}\left({}^B\mathcal{V}\right) = \lim_{\Delta t \to 0} \frac{{}^B\mathcal{V}(t+\Delta t) - {}^B\mathcal{V}(t)}{\Delta t}$$

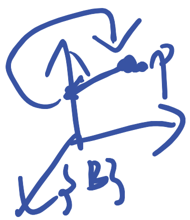
$$\neq {}^B\!\left(\frac{d}{dt}\mathcal{V}\right) = {}^B\!A$$

- ${}^O\!\mathcal{A} \triangleq {}^O\!\left(\frac{d}{dt}\mathcal{V}\right)$ and ${}^B\!\mathcal{A} \triangleq {}^B\!\left(\frac{d}{dt}\mathcal{V}\right)$
- In general, ${}^B\!\mathcal{A} \neq \frac{d}{dt}\left({}^B\mathcal{V}\right)$
- Fact: ${}^O\!\mathcal{A} = {}^O X_B \, {}^B\!\mathcal{A}$ (more about this later)

More about Conventional Vector Cross Product

- Consider the inertia frame {O} and a rotating frame {B} with collocated origins. Let $R(t) \triangleq {}^B R_b(t)$ be the orientation of {B} wrt {O}

- Suppose {B} is rotating with velocity ω at time t. Consider a point p rigidly attached to {B}.



$$\begin{cases} {}^O p = R\, {}^B p \Rightarrow {}^O\dot{p} = \dot{R}\, {}^B p \\ {}^O\dot{p} = {}^O\omega \times R\, {}^B p \end{cases} \quad \Rightarrow [\omega] = \dot{R}R^{-1}$$

$$\dot{R} = [\omega]R \quad \Rightarrow [\omega] = \dot{R}R^{-1}$$

- Therefore, for a rotating frame with time-varying orientation R, its *instantaneous* angular velocity is given by $\dot{R}R^{-1}$

- One can also show (algebraically)
  - $[R\omega] = R[\omega]R^T \Leftarrow$
  - $[\underbrace{\omega_1 \times \omega_2}_{\mathbb{R}^3}] = [\omega_1][\omega_2] - [\omega_2][\omega_1] \Leftarrow$ Jacobi's identity

Spatial Cross Product

- Again, consider a point p rigidly attached to rotating frame (with angular velocity ω), then $\dot{p} = \omega \times p$
- Cross product can be viewed as a differentiation operator. This can be generalized to spatial vectors, leading to *spatial cross product*
- Given two spatial velocities (twists) $\mathcal{V}_1$ and $\mathcal{V}_2$, their spatial cross product is:

$$\mathcal{V}_1 \times \mathcal{V}_2 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \times \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}$$

- Matrix representation: $\mathcal{V}_1 \times \mathcal{V}_2 = [\mathcal{V}_1 \times]\mathcal{V}_2$, where

$$[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}$$

- Roughly speaking, when a motion vector $\mathcal{V}$ is moving with a spatial velocity $\mathcal{Z}$ (e.g. it is attached to a moving frame) but is otherwise not changing, then

$$\dot{\mathcal{V}} = \mathcal{Z} \times \mathcal{V}$$

Spatial Cross Product: Properties (1/2)

- Assume A is moving wrt to O with velocity $\mathcal{V}_A$

$$^O\dot{X}_A = [^O\mathcal{V}_A\times]^O X_A$$

If $^O T_A = (R, p)$, $^O X_A = [Ad_{T_A}] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$

$$^O\dot{X}_A = \begin{bmatrix} \dot{R} & 0 \\ \frac{d}{dt}[p]R + [p]\dot{R} & \dot{R} \end{bmatrix} = \begin{bmatrix} [\omega]R & 0 \\ [v_0]R + [\omega][p]R & [\omega]R \end{bmatrix} = \underbrace{\begin{bmatrix} [\omega] & 0 \\ [v_0] & [\omega] \end{bmatrix}}_{[^O\mathcal{V}_A\times]} X_A$$

$$[\dot{p}] = [v_0] + [\omega\times p] = [v_0] + [\omega][p] - [p][\omega]$$

$$\dot{p} = v_0 + \omega\times p$$

$$\Rightarrow [\dot{p}]R + [p]\dot{R} = [v_0]R + [\omega][p]R - [p][\omega]R + [p]\dot{R} \quad \dot{R} \to [\omega]R$$

$$= [v_0]R + [\omega][p]R$$

Spatial Cross Product: Properties (2/2)

- $\underbrace{[X\mathcal{V}\times]}_{\Downarrow} = \underbrace{X[\mathcal{V}\times]X^T}$, for any transformation X and twist $\mathcal{V}$

$[R\omega] = R[\omega]R^T$

Spatial Cross Product: Working with Moving Frame

Consider a body with velocity $\mathcal{V}_{body}$ (wrt inertia frame), and ${}^O\mathcal{V}_{body}$ and ${}^B\mathcal{V}_{body}$ be its Plücker coordinates wrt $\{O\}$ and $\{B\}$:

- ${}^B\mathcal{A}_{body} = \underbrace{\frac{d}{dt}\left({}^B\mathcal{V}_{body}\right)}_{\text{apparent derivative}} + {}^B\mathcal{V} \times {}^B\mathcal{V}_{body}$ ← due to frame $\{B\}$ is moving

$\triangleq {}^B\left(\frac{d\,\mathcal{V}_{body}}{dt}\right)$

${}^O\mathcal{A}_{body} = \frac{d}{dt}\left({}^O\mathcal{V}_{body}\right) + {}^O\mathcal{V} \times {}^O\mathcal{V}_{body} = \frac{d}{dt}\left({}^O\mathcal{V}_{body}\right)$

we know ${}^O\mathcal{V}_{body} = {}^OX_B\, {}^B\mathcal{V}_{body}$

- $\boxed{{}^O\mathcal{A} = {}^OX_B\, {}^B\mathcal{A}}$

$${}^O\mathcal{A}_{body} = \frac{d}{dt}\left({}^O\mathcal{V}_{body}\right) = \frac{d}{dt}\left({}^OX_B\, {}^B\mathcal{V}_{body}\right)$$

$$= {}^O\dot{X}_B\, {}^B\mathcal{V}_{body} + {}^OX_B\, {}^B\dot{\mathcal{V}}_{body}$$

$$= \left[{}^O\mathcal{V}_B \times\right] {}^OX_B\, {}^B\mathcal{V}_{body} + {}^OX_B\, {}^B\dot{\mathcal{V}}_{body}$$

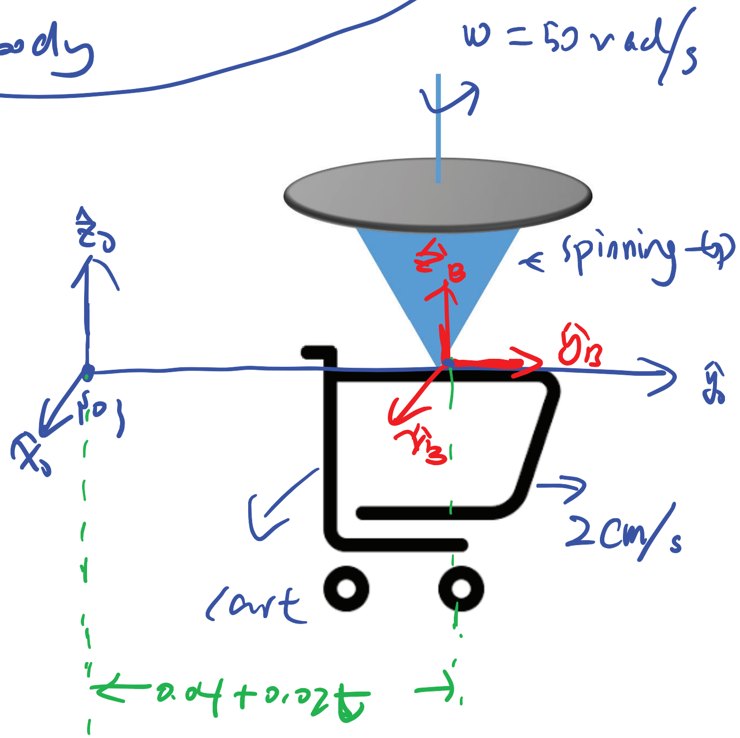
$$= {}^OX_B \left\{ {}^BX_O \left[{}^O\mathcal{V}_B \times\right] {}^OX_B\, {}^B\mathcal{V}_{body} + {}^B\dot{\mathcal{V}}_{body} \right\} =$$

$$= \qquad \left[{}^BX_O\, {}^O\mathcal{V}_B \times\right] = \left[{}^B\mathcal{V}_B \times\right] \leftarrow$$

Spatial Acceleration Example I

$$= {}^0X_B \{ \underbrace{{}^B\dot{\mathcal{V}}_{body} + {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body}}_{{}^BA_{body}} \}$$

Find ${}^BA_{top}$



Method 1: ${}^BA_{top} = {}^BX_0 \, {}^0A_{top} = {}^BX_0 \, {}^0\dot{\mathcal{V}}_{top}$

$${}^0\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ \hdashline 50\cdot(0.04+0.02t) \\ 0.02 \\ 0 \end{bmatrix}, \quad {}^0A_{top} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$${}^BA_{top} = {}^BX_0 \, {}^0A_{top} = \begin{bmatrix} I & 0 \\ [{}^Bp_0] & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Method 2:

$${}^B\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix}$$

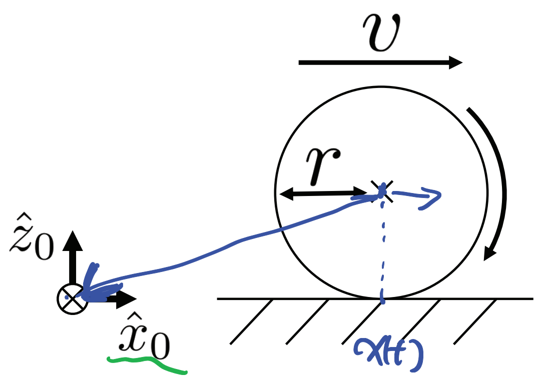
$${}^BA_{top} = \underbrace{\frac{d}{dt}({}^B\mathcal{V}_{top})}_{0} + {}^B\mathcal{V}_{Bframe} \times {}^B\mathcal{V}_{top}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Spatial Acceleration Example II

- A cylinder rolls without slipping in the $\hat{x}_0$ direction. The cylinder has a radius of r and a constant forward speed of v. What is the spatial acceleration of this cylinder? Find ${}^0\mathcal{A}_{cyl}$

$${}^0\mathcal{V}_{cyl} = \begin{bmatrix} {}^0\omega \\ {}^0v_0 \end{bmatrix}, \quad {}^0\omega = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix}$$



$${}^0v_0 = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \omega \times \begin{bmatrix} -x(t) \\ 0 \\ -r \end{bmatrix}$$

$$x(t) = \bar{x}_0 + v\cdot t$$

$$= \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix} \times \begin{bmatrix} -(\bar{x}_0 + v\cdot t) \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{v}{r} \\ 0 & 0 & 0 \\ -\frac{v}{r} & 0 & 0 \end{bmatrix} \begin{bmatrix} -(\bar{x}_0 + v\cdot t) \\ 0 \\ -r \end{bmatrix} = \begin{bmatrix} v - v \\ 0 \\ \frac{v}{r}(\bar{x}_0 + vt) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{v}{r}(\bar{x}_0 + vt) \end{bmatrix}$$

$$\Rightarrow \quad {}^0\mathcal{A}_{cyl} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{v^2}{r} \end{bmatrix}$$

Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Force (Wrench)

screw . wrench

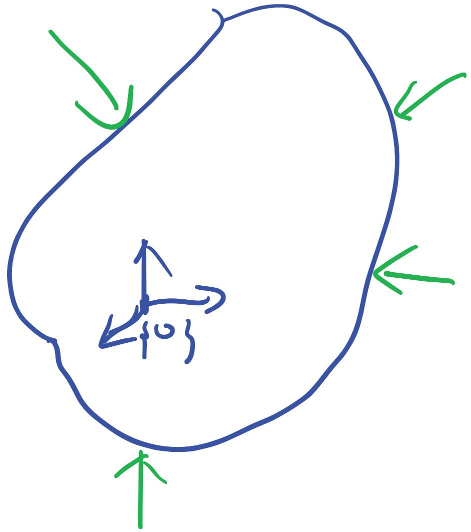
- Consider a rigid body with many forces on it and fix an arbitrary point O in space

- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O

    $f = \sum f_i \in \mathbb{R}^3$

  - A moment n_O about point O

    $n_O \in \mathbb{R}^3$



- **Spatial Force (Wrench):** is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

Spatial Force in Plücker Coordinate Systems

- Given a frame $\{A\}$, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

$$^B\mathcal{F} = \begin{bmatrix} ^Bf \\ ^Bn_{O_B} \end{bmatrix} \qquad ^A\mathcal{F} = \begin{bmatrix} ^An_{O_A} \\ ^Af \end{bmatrix}$$

- Coordinate transform: $^A\mathcal{F} = {}^AX_B^*\,{}^B\mathcal{F}$ where $\underline{^AX_B^* = {}^BX_A{}^T}$

$$\boxed{^Af = {}^AR_B\,{}^Bf} \checkmark$$

moment relation: coordinate free: $n_{O_A} = n_{O_B} + \vec{AB} \times f$

choose {A} frame to state the relation:

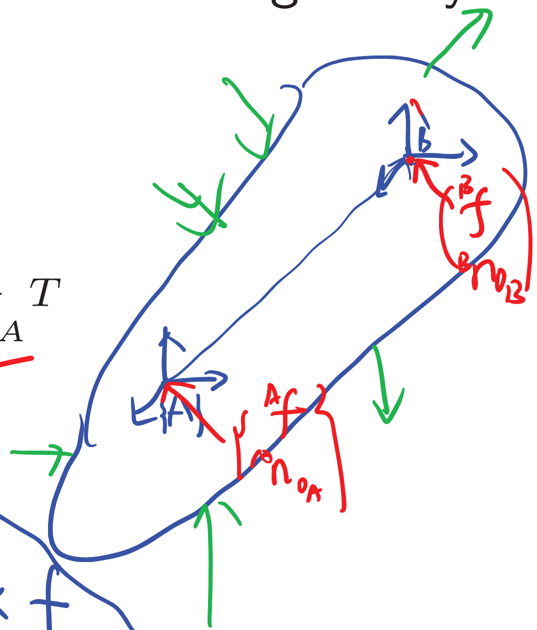
$$^An_{O_A} = {}^An_{O_B} + {}^A(\vec{AB}) \times {}^Af$$

$$= {}^AR_B\,\underline{^Bn_{O_B}} + {}^AR_B(-\underline{^Bp_A \times {}^Bf})$$

$$= {}^AR_B\,{}^Bn_{O_B} - {}^AR_B[{}^Bp_A]\,{}^Bf$$

$$\Rightarrow \begin{bmatrix} ^An_{O_A} \\ ^Af \end{bmatrix} = \underbrace{\begin{bmatrix} ^AR_B & -{}^AR_B[{}^Bp_A] \\ 0 & ^AR_B \end{bmatrix}}_{\triangleq\, ^AX_B^*} \begin{bmatrix} ^Bn_{O_B} \\ ^Bf \end{bmatrix}$$

$$^AX_B^* = ({}^BX_A)^T$$



Wrench-Twist Pair and Power

$f \in \mathbb{R}^3$

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

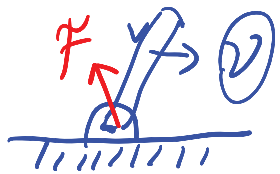
$$P = ({}^A\mathcal{V})^T \; {}^A\mathcal{F}$$

indep of ref frame

$$= \underbrace{({}^A\omega)^T \cdot {}^A n_{o_A}}_{\text{rotational power}} + ({}^A v_{o_A})^T \cdot {}^A f$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by: $\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$



- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \underbrace{\mathcal{V}^T}_{\hat{S}^T\dot{\theta}}\mathcal{F} = \underbrace{(\hat{\mathcal{S}}^T\mathcal{F})}_{\tau}\dot{\theta} \triangleq \underbrace{\tau\dot{\theta}}_{\text{scalar}}$$

- $\tau = \hat{\mathcal{S}}^T\mathcal{F} = \mathcal{F}^T\hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

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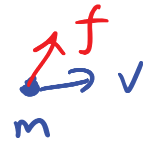
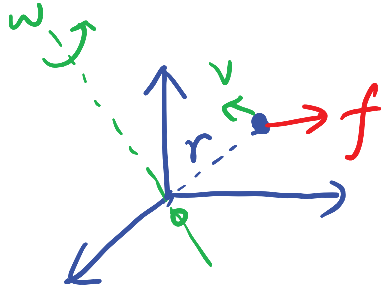
Rotational Inertia (1/2)





- Recall momentum for point mass:

velocity: $v = \dot{r}$, $a = \dot{v} = \ddot{r} \in \mathbb{R}^3$

force: $f = ma = m\dot{v} = m\ddot{r} \in \mathbb{R}^3$ $\longleftrightarrow$ moment: $n = r \times f$

$\omega = \hat{\omega}\dot{\theta}$, $v = \omega \times r$

$\begin{pmatrix} a \times b \\ = -b \times a \end{pmatrix}$

Linear momentum: $L = m \cdot v$ $\longleftrightarrow$ angular momentum: $\phi = r \times L$

$$= r \times (m \cdot \omega \times r)$$

$$= m \cdot (r \times \omega \times r)$$

$$= m\,(r \times (-r) \times \omega)$$

$$= \underbrace{(m\,[r]\,[-r])}_{\substack{\text{Inertia matrix} \\ 3\times 3 \text{ matrix}}}\,\omega \rightarrow \mathbb{R}^3$$

$$= m\,[r]\,[r]^T \omega$$

$$f = \frac{d}{dt} L$$

Linear motion

Rotational Inertia (2/2)

$\approx \sum_i m_i [r_i][r_i]^T$

- Rotational Inertia: $\bar{I} = \int_V \rho(r)[r][r]^T dr$
  - $\rho(\cdot)$ is the density function of the body
  - $\bar{I}$ depends on coordinate system
  - It is a constant matrix if the origin coincides with CoM

frame origin

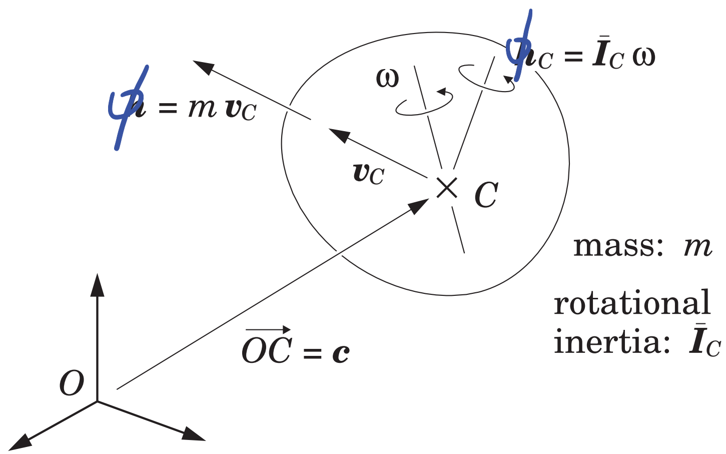
${}_C\bar{I}$: rotational inertia matrix about CoM

Spatial Momentum $\leftarrow$

- Consider a rigid body with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C
  - Linear momentum: $\underbrace{L_c = m v_c}$ $\qquad \sum m_i v_c$
  - Angular momentum about CoM: $\underbrace{\phi_c}_{3\times 1} = \underbrace{\bar{I}_c}_{3\times 3} \underbrace{\omega}_{3\times 1}$
  - Angular momentum about a point O: $\rightarrow \phi_o = \phi_c + \vec{OC} \times L_c \Leftarrow$ coordinate free

- Spatial Momentum:

$$h \triangleq \begin{bmatrix} \phi \\ L \end{bmatrix} \in \mathbb{R}^6$$



Change Reference Frame for Momentum

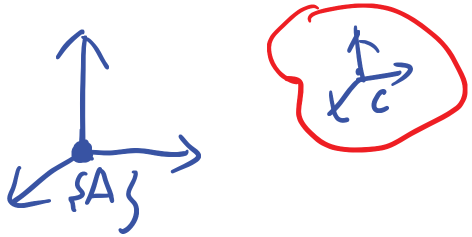
- Spatial momentum transforms in the same way as spatial forces:

$$^Ah = {}^AX_C^* \, {}^Ch$$

$$^AX_C^* \rightarrow ({}^CX_A)^T$$

$$^Ch = \begin{bmatrix} ^C\phi_C \\ ^CL \end{bmatrix}$$

$$^Ah = \begin{bmatrix} ^A\phi_A \\ ^AL \end{bmatrix}$$



$$\boxed{\underset{3\times 1}{^AL} = \underset{3\times 3}{^AR_C} \; \underset{3\times 1}{^CL}}$$

$$\phi_A = \phi_C + \vec{AC} \times L \Rightarrow {}^A\phi = {}^AR_C \, {}^C\phi + {}^AR_C(-{}^Cp_A \times {}^CL)$$

$$\Rightarrow \quad {}^Ah = {}^AX_C^* \, {}^Ch$$

Spatial momentum transforms the same as spatial forces

Spatial Inertia

- Inertia of a rigid body defines linear relationship between velocity and momentum.

- Spacial inertia $\mathcal{I}$ is the one such that

6x6

$$\underbrace{h}_{6\times 1} = \mathcal{I}\underbrace{\mathcal{V}}_{6\times 1}$$

- Let {C} be a frame whose origin coincide with CoM. Then

constant 3x3 matrix

$${}^C\mathcal{I} = \begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

3x3 identity matrix

$${}^C h = \begin{bmatrix} {}^C\bar{I}_c \omega \\ m\,{}^C v \end{bmatrix} = \underbrace{\begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}}_{{}^C\mathcal{I}} \begin{bmatrix} {}^C\mathcal{V} \end{bmatrix}$$

Spatial Inertia

- Spatial inertia wrt another frame $\{A\}$:

$$^{A}\mathcal{I} = {}^{A}X_C^{*}\, {}^{C}\mathcal{I}\, {}^{C}X_A$$

$$^{A}h = {}^{A}\mathcal{I}\, {}^{A}\mathcal{V} = {}^{A}X_C^{*}\, {}^{C}h = {}^{A}X_C^{*}\, {}^{C}\mathcal{I}\, {}^{C}\mathcal{V} = {}^{A}X_C^{*}\, {}^{C}\mathcal{I}\, {}^{C}X_A\, {}^{A}\mathcal{V}$$

- Special case: $^{A}R_C = I_3$ (no change on orientation)

$$^{A}X_C = \begin{bmatrix} I & 0 \\ [{}^{A}p_c] & I \end{bmatrix} \Rightarrow {}^{A}\mathcal{I} = \begin{bmatrix} {}^{C}\mathcal{I} + m[{}^{A}p_c][{}^{A}p_c]^T & m[{}^{A}p_c] \\ m[{}^{A}p_c] & m I_{3\times 3} \end{bmatrix}$$

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- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Cross Product for Spatial Force and Momentum

Assume frame A is moving with velocity ${}^{A}\mathcal{V}_A$

- $\underbrace{{}^{A}\left[\frac{d}{dt}\mathcal{F}\right]}_{\text{coordinate free expression}} = \underbrace{\frac{d}{dt}\left({}^{A}\mathcal{F}\right)}_{\text{apparent derivative}} + {}^{A}\mathcal{V}\times^{*}{}^{A}\mathcal{F}$

- ${}^{A}\left[\frac{d}{dt}h\right] = \frac{d}{dt}\left({}^{A}h\right) + {}^{A}\mathcal{V}\times^{*}{}^{A}h$

$$\mathcal{V}\times^{*}\mathcal{F} = \begin{bmatrix} w\times n + v\times f \\ w\times f \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} w \\ v \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} n \\ f \end{bmatrix}$$

$$\underbrace{[\mathcal{V}\times^{*}]}_{6\times 6} = -[\mathcal{V}\times]^{T}$$

cross product for spatial force and momentum

" $\times^{*}$ "

$$\frac{d}{dt}\left({}^{0}X_A^{*}\right) = \underbrace{\left[{}^{0}\mathcal{V}_A\times^{*}\right]}_{6\times 6}{}^{0}X_A^{*}$$

$$\boxed{\frac{d}{dt}\left({}^{0}X_A\right) = \left[\mathcal{V}_A\times\right]{}^{0}X_A}$$

Newton-Euler Equation

$${}^A\mathcal{F} = {}^A\mathcal{I}\,{}^A\mathcal{A} + {}^A\mathcal{V}\times^* ({}^A\mathcal{I}\,{}^A\mathcal{V})$$

$${}^B\mathcal{F} = {}^B\mathcal{I}\,{}^B\mathcal{A} + {}^B\mathcal{V}\times^* ({}^B\mathcal{I}\,{}^B\mathcal{V})$$

- Newton-Euler equation:

$$\mathcal{F} = \frac{d}{dt}h = \mathcal{I}\mathcal{A} + \mathcal{V}\times^* \mathcal{I}\mathcal{V}$$

← coordinate free

- Adopting spatial vectors, the Newton-Euler equation has the same form in any frame

origin at com

- Choose an ~~arbitrary~~ frame $\{A\}$, let ${}^A\mathcal{V} = ({}^A\omega, {}^Av)$ ~ Details: see discussion on page 31.
  - Newton equation:

$$ {}^Af = m\,{}^A\dot{v} + {}^A\omega \times m\,{}^Av$$

    - Note: if $\{A\}$ is inertia frame $\{o\}$, we have

$$ {}^of = m\,{}^o\dot{v} + {}^o\omega \times m\,{}^ov = m\,{}^o\ddot{p}_{com}$$

  - Euler equation:

$$ {}^An = {}^A\bar{I}\,{}^A\dot{\omega} + {}^A\omega \times {}^A\bar{I}\,{}^A\omega$$

Derivations of Newton-Euler Equation

$^0\mathcal{V}$: is velocity of body in $\{0\}$

- choose inertia frame $\{0\}$: $\quad {}^0f = \frac{d}{dt}({}^0h)$

$$\frac{d}{dt}({}^0h) = \frac{d}{dt}({}^0\mathcal{I}\,{}^0\mathcal{V}) = \frac{d}{dt}\big(\underbrace{{}^0X_B^*\,{}^B\mathcal{I}\,{}^BX_0}_{{}^0\mathcal{I}}\,{}^0\mathcal{V}_B\big)$$

- suppose $\{B\}$ is body frame

${}^B\mathcal{I}$ is constant

- ${}^B\mathcal{V}$: velocity of $\{B\}$

${}^0\mathcal{V}_B$

$$= {}^0\dot{X}_B^*\,{}^B\mathcal{I}\,{}^BX_0\,{}^0\mathcal{V}_B + {}^0X_B^*\,{}^B\mathcal{I}\,{}^B\dot{X}_0\,{}^0\mathcal{V}_B + {}^0X_B^*\,{}^B\mathcal{I}\,{}^BX_0\,{}^0\dot{\mathcal{V}}_B$$

$$= [{}^0\mathcal{V}_B\times^*]\,\underbrace{{}^0X_B^*\,{}^B\mathcal{I}\,{}^BX_0}_{{}^0\mathcal{I}}\,{}^0\mathcal{V}_B - {}^0X_B^*\,{}^B\mathcal{I}\,{}^BX_0\,\underbrace{[{}^0\mathcal{V}_B\times]\,{}^0\mathcal{V}_B}_{=0} + {}^0\mathcal{I}\,{}^0\mathcal{A}$$

$$\boxed{= {}^0\mathcal{I}\,{}^0\mathcal{A} + {}^0\mathcal{V}_B\times^*\,{}^0\mathcal{I}\,{}^0\mathcal{V}_B}$$

$\dot{*}$, $\dot{X}$,

Note: $\boxed{{}^0\dot{X}_B = [{}^0\mathcal{V}_B\times]\,{}^0X_B}$

$\Rightarrow\ {}^B\dot{X}_0 = -{}^BX_0[{}^0\mathcal{V}_B\times]$

$({}^0X_B\cdot{}^BX_0)' = (I)' = 0$

${}^0\dot{X}_B\,{}^BX_0 + {}^0X_B\,{}^B\dot{X}_0 \Rightarrow {}^B\dot{X}_0 = -{}^BX_0\,{}^0\dot{X}_B\,{}^BX_0$

$= -{}^BX_0[{}^0\mathcal{V}_B\times]\,\cancel{{}^0X_B}\,{}^BX_0$

More Discussions : relation to classical equations of motion :

- Pick arbitrary frame $\{A\}$: Newton Euler equation:

$$\underbrace{\begin{bmatrix} {}^A n_A \\ {}^A f \end{bmatrix}}_{{}^A\mathcal{F}} = \underbrace{\left({}^A X_c^* \begin{bmatrix} {}^c\bar{I} & 0 \\ 0 & m I_3 \end{bmatrix} {}^A X_c\right)}_{{}^A\mathcal{I}} \cdot \underbrace{\begin{bmatrix} {}^A\dot{\omega} \\ {}^A\dot{v}_A \end{bmatrix}}_{{}^A\mathcal{A}} + \underbrace{\begin{bmatrix} [{}^A\omega] & [{}^A v_A] \\ 0 & [{}^A\omega] \end{bmatrix}}_{{}^A\mathcal{V}\times^*} \underbrace{{}^A X_c^* \begin{bmatrix} {}^c\bar{I} & 0 \\ 0 & m I_3 \end{bmatrix} {}^A X_c}_{{}^A\mathcal{I}} \underbrace{\begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix}}_{{}^A\mathcal{V}}$$

If $\{A\}$ is CoM frame : then ${}^A X_c^* = {}^A X_c = I$

$\{A\} = \{C\}$

$$\Rightarrow \begin{bmatrix} {}^c n_c \\ {}^c f \end{bmatrix} = \begin{bmatrix} {}^c\bar{I} & 0 \\ 0 & m I_3 \end{bmatrix} \begin{bmatrix} {}^c\dot{\omega} \\ {}^c\dot{v}_c \end{bmatrix} + \begin{bmatrix} [{}^c\omega] & [{}^c v_c] \\ 0 & [{}^c\omega] \end{bmatrix} \begin{bmatrix} {}^c\bar{I} & 0 \\ 0 & m I \end{bmatrix} \begin{bmatrix} {}^c\omega \\ {}^c v_c \end{bmatrix}$$

$$= \begin{bmatrix} {}^c\bar{I}\,{}^c\dot{\omega} \\ m\,{}^c\dot{v}_c \end{bmatrix} + \begin{bmatrix} {}^c\omega \times ({}^c\bar{I}\,{}^c\omega) + \underbrace{m\,{}^c v_c \times v_c}_{0} \\ m\,{}^c\omega \times {}^c v_c \end{bmatrix}$$

$$
= \begin{bmatrix} {}^c I \, {}^c\dot{w} + {}^c w \times {}^c I \, {}^c w \\ \\ m \, {}^c\dot{v}_c + m \, {}^c w \times {}^c v_c \end{bmatrix}
$$

Note: ${}^c\dot{v}_c \neq {}^c\ddot{p}_{com}$ ${}^c v_c = {}^c\dot{p}_{com}$

by early discussion we know

$$
{}^c\dot{v}_c = {}^c\ddot{p}_{com} - {}^c w \times \underbrace{{}^c\dot{p}_{com}}_{{}^c v_c}
$$

$$
= \begin{bmatrix} {}^c I \, {}^c\dot{w} + {}^c w \times {}^c I \, {}^c w \\ \\ m \, {}^c\ddot{p}_{com} \end{bmatrix}
$$

$\leftarrow$ Euler Equ

$\leftarrow$ Newton Equ.

More Discussions

- Spacial velocity: ${}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^Bv_{o_B} \end{bmatrix}$

- spacial acceleration: $\mathcal{A}_{body} = \dot{\mathcal{V}}_{body}$ (coordinate free)

$$ {}^0\mathcal{A}_{body} = \frac{d}{dt}\left({}^0\mathcal{V}_{body}\right) $$

$$ {}^B\mathcal{A}_{body} = \frac{d}{dt}\left({}^B\mathcal{V}_{body}\right) + {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body} $$

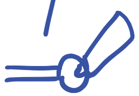
$$ {}^B\mathcal{A}_{body} = {}^BX_0 \; {}^0\mathcal{A}_{body} $$

- $\dot{R}_A = \omega_A \times R_A = \overset{3\times 3}{[\omega_A]} R_A$, $[R\omega] = R[\omega]R^T$

$$ {}^0\dot{X}_A = {}^0\mathcal{V}_A \times {}^0X_A = \underbrace{[{}^0\mathcal{V}_A \times]}_{6\times 6} {}^0X_A \quad , \quad [\underbrace{X\mathcal{V}}_{6\times 1}\times] = X[\mathcal{V}\times]X^T $$

- spatial force (wrench): ${}^B\mathcal{F} = \begin{bmatrix} {}^Bn \\ {}^Bf \end{bmatrix}$, ${}^A\mathcal{F} = {}^AX_B^* \, {}^B\mathcal{F}$, ${}^AX_B^* = ({}^BX_A)^T$

- Joint torque: $\tau \cdot \dot{\theta} = \mathcal{V}^T\mathcal{F}$



$$ \tau = S^T\mathcal{F} $$ ← for single joint

- spatial momentum: $\quad {}^A h = \begin{bmatrix} {}^A\phi_A \\ {}^A L \end{bmatrix}$, $\quad {}^A h = {}^A X_B^* \; {}^B h$

- spatial inertia:

$$\underset{\text{com}}{{}^C} \underset{\sim}{I} = \left[\begin{array}{c|c} {}^C \underset{\sim}{\bar{I}} & \underset{-}{0} \\ \hline 0 & m I_3 \end{array}\right] \qquad {}^B \underset{\sim}{I} = \left({}^B X_C^*\right) {}^C \underset{\sim}{I} \left({}^B X_C\right)$$