MEE524 Modern Control and Estimation

LN1: Linear Algebra Review

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Why Linear Algebra:

- One of the most important tools for modern control theory

- Topics covered in this class, such as
  - State space model
  - Least squares
  - Stability analysis
  - Controller/observer design through eigenvalue assignment
  - Linear quadratic regulator (LQR)
  - Kalman filter

  can be viewed as applications of linear algebra

- Crucial for machine learning, robotics, computer vision, …

Facts about the students:

- Remember formulas without deep understanding of concepts
- Good at numerical calculations, but not analytical analysis

Goal:

- Not to recall the formulas or numerical techniques
- Review/rebuild fundamental concepts
- "Speak" the language of linear algebra: formulate/analyze/solve linear algebra problems without using formulas or numbers
  - Linear independence
  - Matrix rank
  - Vector space
  - Column space/null space
  - Solution $Ax = b$

- Just a short review. A good reference is the MIT course (Prof. Strang)

https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/index.htm

Review Outline

- **Part I:**
  - **Linear combination**
  - **Linear independence**
  - **Vector space**
- Part II:
  - Column space/Null space
  - Matrix rank
  - Solution space of $Ax = b$
- Part III
  - Inner product
  - Simple geometric sets
  - Quadratic sets

Key Concept: Linear Combination

- Linear combination of two vectors $v_1, v_2 \in R^2$

- Linear combination of $v_1, v_2, \ldots, v_k \in R^n$

Key Concept: Linear Combination

- Trivial and nontrivial linear combination:

- Span of a set of vectors:

  $span(v_1, v_2, \ldots, v_k) = \{w \in R^n : w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k, \text{for some scalars } \alpha_1, \alpha_2, \ldots, \alpha_k\}$

Linear Independence

- Two vectors $\{v_1, v_2\}$ are **linearly dependent** if

- A set of vectors $\{v_1, \cdots, v_k\}$ is linearly **independent** if

No nontrivial linear combination = 0

- Equivalent definition: No vector v_i can be expressed as a linear combination of other vectors $v_1, \cdots, v_{i-1}, v_{i+1}, \dots, v_n$

Vector Space

- **Vector space $\boldsymbol{V}$**: set of elements for which "addition" and "multiplication by scalers" can be properly defined
  - element can be number, matrix, function, symbols …
  - "Addition" and "multiplication" can be defined as long as they satisfy certain Axioms.

- **Subspace** of a vector space V: subset of V that is "closed" under addition and multiplication
  - Span (v_1, v_2) :
  - $R_+^2 = \{x \in R^2 : x_1 \geq 0, x_2 \geq 0\}$:

Vector Space

- $\{v_1, v_2, \dots, v_k\}$ is a basis for a vector space V if
  1. $V = span(\{v_1, v_2, \dots, v_k\})$
  2. $\{v_1, v_2, \dots, v_k\}$ is linearly independent

- Dimension of a vector space:
  - Number of vectors in a basis
  - **Fact:** number of vectors in any basis of a finite-dim vector space is the same

Vector Space

- Coordinates of $w \in V$ with respect to a basis $\{v_1, v_2, \dots, v_k\}$

- Coordinates of a vector depend on the basis

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Key: Matrix vector multiplication as mixture of columns

- Let $y = Ax$, then y is a linear combination of the columns of A
  - Write matrix A in terms of its columns

$$A = [a_1 \quad a_2 \quad \dots \quad a_n], \text{ where } a_j \in R^m$$

  - Then $y = Ax$ can be written as

- Similarly, if $z = dB$, where d and z are row vectors, then z is a linear combination of the rows of B

Column Space (Range) of a Matrix $A \in R^{m \times n}$

$$\text{Col(A)} = \text{Range}(A) = \{ Ax \mid x \in R^n \} \subset R^m$$

- **= Span of columns of *A***

- Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

Null Space of a Matrix $A \in R^{m \times n}$

$$Null(A) = \{x \in R^n | Ax = 0\}$$

- Coefficients of linear combinations that result in a zero vector
- Zero null space implies: columns of A are independent
- Example of null space: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

Rank of a matrix $A \in R^{m \times n}$

- Definition: $rank(A) = \dim(\mathrm{Col}(A))$
  - i.e. number of independent columns of A

- Nontrivial facts
  - $rank(A) = \; rank(A^T)$
  - $rank(A) \leq \min(m, n)$: full rank means **rank(A) = min(m,n)**
  - $rank(A) + \dim\big(Null(A)\big) = \; n$
    - "**conservation of dimension**": Think about A as a linear mapping that maps $x \in R^n$ to a vector $y = Ax \in R^m$. Each dimension of input x is either crushed to zero or ends up in output

Example of "conservation of dimension":

- Find the $Null(A)$, where $A \in R^{10\times 4} = [a_1 \;\; a_2 \;\; a_3 \;\; a_4]$ satisfies

① $a_2 = 2a_1 + 5a_3$

② $a_4 = 5a_1 - 6a_3$

Linear Equation $Ax = b,\ x \in R^n, b \in R^m$

- There exists a solution if

- There always exists a solution for any $b \in R^m$ if:

- There exists a unique solution if:

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Geometry and linear algebra

- Points , angles, lines, planes, convex sets, cones, polytopes, balls, ellipsoids

- Numerical representation of the same geometrical or physical quantity changes with the coordinate system

- Most of them can be generalized to R^n or even general vector space

Key Geometric Quantity: Inner Product

- Inner product $<\cdot,\cdot>: V \times V \to R$:
  - maps each pair in a vector space to a scaler
  - Satisfies several key properties: linearity, conjugate, positive definiteness …
- Inner product of vectors in R^n: $< v, w > =$
- Inner product of matrices in $R^{m \times n}$: $< A, B > =$
- Inner product of two functions f, g on interval $[a, b]$:

Key Geometric Quantity: Inner Product

- Inner product defines important geometric notions:
  - Norm:
  - Angle
  - Orthogonality

Some Simple Geometric Sets

- Line and line segment: given $x_1 \neq x_2 \in R^n$:

$$y = \theta x_1 + (1 - \theta) x_2$$

- Hyperplanes: $\{x: a^T x = b\}$

- Halfspaces: $\{x: a^T x \leq b\}$

Some Simple Geometric Sets

- Convex set: A set S is called convex if

$$x_1, x_2 \in S \Rightarrow \alpha x_1 + (1-\alpha) x_2 \in S$$

- Convex combination of $x_1, \ldots, x_k \in R^n$

$$\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k : \alpha_i \geq 0, \sum \alpha_i = 1\}$$

- Convex hull $\overline{co}(S)$: set of all convex combinations of points in S

Some Simple Geometric Sets

- **Cone**: A set S is called a cone if $x \in S \Rightarrow \lambda x \in S, \ \forall \lambda \geq 0$

- **Conic combination** of $x_1, \dots, x_k \in R^n$

$$cone(x_1, \dots, x_k) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \colon \alpha_i \geq 0\}$$

Some Simple Geometric Sets

- **(Convex) Polyhedron**: intersection of a finite number of half spaces

$$P = \{x : Ax \leq b\}$$

- **Polyhedral cone**: intersection of finitely many halfspaces that contain the origin:

$$P = \{x : Ax \leq 0\}$$

- **Polytope**: bounded polyhedron

Some Simple Geometric Sets

- Polyhedron: intersection of a finite number of half spaces $P = \{x : Ax \leq b\}$

- Polyhedron (vertex representation):

$$P = \overline{co}(v_1, \dots, v_m) \oplus cone(r_1, \dots, r_q)$$

Symmetric Matrix

- Symmetric: $A = A^T$

- Key property: **Spectral decomposition**

$$A \text{ is symmetric } \Leftrightarrow A = Q\Lambda Q^T, \text{ where } A \text{ is diagonal and } Q \text{ is unitary}$$

Positive Semi-definite Matrices

- Positive semidefinite: $x^T A x \geq 0, \ \forall x \in R^n$

- PSD matrices define sign-definite quadratic forms:
  - Examples:

Positive Semi-definite Matrices

- Equivalent Definitions of PSD Matrices:
  - All $2^{n} - 1$ leading principal minors are nonnegative
  - All eigs are nonnegative
  - There exists a factorization $A = B^T B$ with B square and nonsingular
- If P is positive definite, then P^{-1} is also positive definite

- Euclidean balls: $B(x_c, r) = \{x: \|x - x_c\|_2 \leq r\}$
  or $B(x_c, r\ ) = \{x_c + ru :\ \|u\|_2 \leq 1\}$

- Euclidean ellipsoids: $E = \{x: (x - x_c)^T P^{-1}(x - x_c)\}$

or $E = \{x_c + Au :\ \|u\|_2 \leq 1\}$