MEE524 Modern Control and Estimation

LN1: Linear Algebra Review

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Why Linear Algebra:

- One of the most important tools for modern control theory

- Topics covered in this class, such as
  - State space model
  - Least squares
  - Stability analysis
  - Controller/observer design through eigenvalue assignment
  - Linear quadratic regulator (LQR)
  - Kalman filter

  can be viewed as applications of linear algebra

- Crucial for machine learning, robotics, computer vision, …

Facts about the students:

- Remember formulas without deep understanding of concepts
- Good at numerical calculations, but not analytical analysis

Goal:

- Not to recall the formulas or numerical techniques
- Review/rebuild fundamental concepts
- "Speak" the language of linear algebra: formulate/analyze/solve linear algebra problems without using formulas or numbers
  - Linear independence
  - Matrix rank
  - Vector space
  - Column space/null space
  - Solution Ax = b

- Just a short review. A good reference is the MIT course (Prof. Strang)

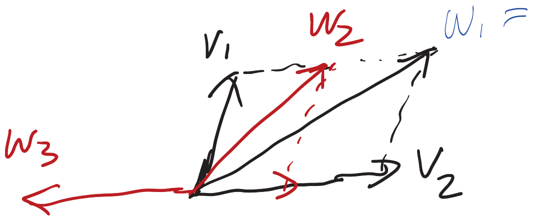
https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/index.htm

Review Outline

- **Part I:**
  - **Linear combination**
  - **Linear independence**
  - **Vector space**
- Part II:
  - Column space/Null space
  - Matrix rank
  - Solution space of Ax = b
- Part III
  - Inner product
  - Simple geometric sets
  - Quadratic sets

Key Concept: Linear Combination

- Linear combination of two vectors $v_1, v_2 \in R^2$



$w_1 = v_1 + v_2$

$w_2 = v_1 + \frac{1}{2} v_2$

$w_3 = - v_2$

$w = \alpha_1 v_1 + \alpha_2 v_2$, for some $\alpha_1, \alpha_2 \in \mathbb{R}$

called a linear combination of v_1, v_2

- Linear combination of $v_1, v_2, \dots, v_k \in R^n$

$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$, for some scalars $\alpha_1, \alpha_2, \cdots \alpha_k \in \mathbb{R}$

Key Concept: Linear Combination

- Trivial and nontrivial linear combination:

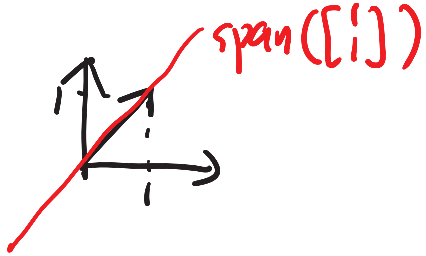
$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ is called $\begin{cases} \text{trivial} & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \\ \text{nontrivial} & \text{otherwise (i.e. at least one } \alpha_i \neq 0) \end{cases}$

- Span of a set of vectors:

$$span(v_1, v_2, \dots, v_k) \triangleq \{w \in R^n : w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k, \text{for some scalars } \alpha_1, \alpha_2, \dots, \alpha_k\}$$

$$= \text{set of all linear combinations of } v_1, \dots, v_k$$

e.g. $span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}\right\}$



$span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \left\{\begin{bmatrix} \alpha \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R}\right\}$

$span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \mathbb{R}^2$

Linear Independence

- Two vectors $\{v_1, v_2\}$ are **linearly dependent** if $v_1 = \alpha v_2$, for some $\alpha \in \mathbb{R}$

  $\Leftrightarrow$ equivalently, there is a nontrivial linear combination $\alpha_1 v_1 + \alpha_2 v_2 = 0$, α_1, α_2 not all zero

- A set of vectors $\{v_1, \cdots, v_k\}$ is linearly **independent** if

  No nontrivial linear combination = 0

  $$\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 \cdots = \alpha_k = 0}$$

  independence

- Equivalent definition: No vector v_i can be expressed as a linear combination of other vectors $v_1, \cdots, v_{i-1}, v_{i+1}, \ldots, v_n$

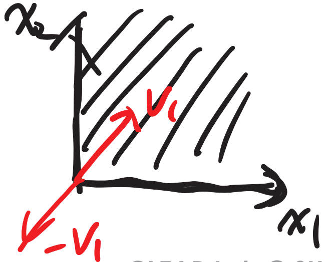
  eg: v_1, v_2, v_3 indep

  $\Rightarrow$ we can't write $v_2 = \alpha_1 v_1 + \alpha_3 v_3$ X

Vector Space

- **Vector space V**: set of elements for which "addition" and "multiplication by scalers" can be properly defined
  - element can be number ($\mathbb{R}$), matrix, function, symbols …
    set of 2x2 matrices
    set of function $f: [0,1] \to \mathbb{R}$
  - "Addition" and "multiplication" can be defined as long as they satisfy certain Axioms.
  - Linear combination is well defined, stays in the space
- **Subspace** of a vector space V: subset of V that is "closed" under addition and multiplication i.e. subset of V for which linear combination stays inside the set
  - subspace must contain "0", because $v \in S$, $0 \cdot v \in S$ ← must belongs to
  - Span $(v_1, v_2) \triangleq \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$ is a subspace

e.g.
- $R_+^2 = \{x \in R^2 : x_1 \geq 0\,, x_2 \geq 0\}$: Not subspace ✗



$v_1 \in \mathbb{R}_+^2$ but $-v_1 \notin \mathbb{R}_+^2$

Vector Space

- $\{v_1, v_2, \dots, v_k\}$ is a basis for a vector space V if
  1. $V = span(\{v_1, v_2, \dots, v_k\})$, for any $v \in V$, v is a ~~a~~ linear combination of $v_1, \dots, v_k$
  2. $\{v_1, v_2, \dots, v_k\}$ is linearly independent

$\mathbb{R}^2$: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is Not a basis $\begin{cases} \text{satisfies 1} \\ \text{Not prop. 2} \end{cases}$

- Dimension of a vector space:
  - Number of vectors in a basis

  - **Fact:** number of vectors in any basis of a finite-dim vector space is the same

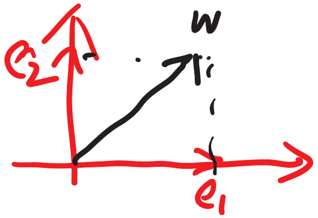
Vector Space

- Coordinates of $w \in V$ with respect to a basis $\{v_1, v_2, \dots, v_k\}$ of V

we must have $w = \alpha_1 v_1 + \cdots + \alpha_k v_k$ for some $\alpha_1 \cdots, \alpha_k$

coordinate of w w.r.t. $\{v_1\ v_2 \dots v_k\}$ is $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$= 1 \cdot e_1 + 1 \cdot e_2$

$= 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Coordinates of a vector depend on the basis

~~change~~ $V = \mathbb{R}^2 = \text{span}\Big(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{e_2}\Big) \Leftarrow w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

change basis: $V = \text{span}\Big(\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_2}\Big) \Leftarrow w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot v_1 + 2 \cdot v_2$

$= [v_1 \,\vdots\, v_2] \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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Key: Matrix vector multiplication as (mixture of columns) linear combination of columns of the matrix

- Let $y = Ax$, then y is a linear combination of the columns of A
  - Write matrix A in terms of its columns

$$A = [a_1 \vdots a_2 \vdots \ldots \ a_n], \text{ where } a_j \in R^m$$

  - Then $y = Ax$ can be written as

$x \in \mathbb{R}^n$

$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$y = Ax = [a_1 \vdots a_2 \vdots \cdots \vdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \cdot a_1 + x_2 \cdot a_2 + \cdots + x_n \cdot a_n$$

scalars

column vectors of A

e.g.: $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Ax = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

$Ax = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$

- Similarly, if $z = dB$, where d and z are row vectors, then z is a linear combination of the rows of B

Column Space (Range) of a Matrix $A \in R^{m \times n}$

$$\text{Col(A)} = \text{Range}(A) = \{ Ax \mid x \in R^n \} \subset R^m$$

$A = [a_1 \cdots , a_n]$

- = **Span of columns of** A $= \text{span}(a_1, \cdots, a_n)$

- Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$

$a_1 + a_3 - a_2 = 0$

$\text{col}(A) = \text{span}(a_1, a_2, a_3)$

$= \text{span}(a_1, a_2)$

$= \text{span}(a_1, a_3)$

Null Space of a Matrix $A \in R^{m \times n}$

$$Null(A) = \{x \in R^n | Ax = 0\}$$ subspace of $\mathbb{R}^n$

; $A \in \mathbb{R}^{10\times 2}$

$Null(A) \subseteq \mathbb{R}^2$

- Coefficients of linear combinations that result in a zero vector
- Zero null space implies: columns of A are independent
- Example of null space: $A = \begin{array}{ccc} \left[\begin{matrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{matrix}\right] \\ a_1 \quad a_2 \quad a_3 \end{array}$, what is Null space

$a_2 = a_1 + a_3 \Rightarrow a_1 - a_2 + a_3 = 0$

$\Rightarrow \underbrace{[a_1 \ a_2 \ a_3]}_{A} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$

$Null(A) = span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$

Rank of a matrix $A \in R^{m\times n}$

- Definition: $rank(A) = \dim(\text{Col}(A))$
  - i.e. number of independent columns of A

- Nontrivial facts
  - $rank(A) = rank(A^T)$, # of indep columns of A = # of indep rows of A
  - $rank(A) \leq \min(m,n)$: full rank means **rank(A) = min(m,n)**

# of indep columns

# of total columns

$A = [\,|\;|\;|\,]$

rank(A) = # of columns

$A = [\,\equiv\,]$

rank(A) = # of rows

  - $rank(A) + \dim(Null(A)) = n$
    - "**conservation of dimension**": Think about A as a linear mapping that maps $x \in R^n$ to a vector $y = Ax \in R^m$. Each dimension of input x is either crushed to zero or ends up in output

$A = [\underbrace{a_1\ a_2\ a_3}_{\text{indep}} \vdots \underbrace{a_4\ \ a_5}_{\text{depends on } \{a_1, a_2, a_3\}}]$

$n=5$

$rank(A)=3$

$\dim(Null(A)) = 2$

Example of "conservation of dimension":

- Find the $Null(A)$, where $A \in R^{10\times 4} = [a_1 \vdots a_2 \vdots a_3 \vdots a_4]$ satisfies $\{a_1, a_3\}$ are indep.

① $a_2 = 2a_1 + 5a_3$

② $a_4 = 5a_1 - 6a_3$

$$\Rightarrow \text{rank}(A) = 2, \quad \Rightarrow n = 4 \quad \Rightarrow \dim(Null(A)) = 4 - 2 = 2$$

To find null space: from ① $\Rightarrow$ $\underbrace{2a_1 - a_2 + 5a_3 + 0\cdot a_4}_{A\begin{bmatrix}2\\-1\\5\\0\end{bmatrix}} = 0$

$$A\begin{bmatrix}2\\-1\\5\\0\end{bmatrix} = 0$$

$\underbrace{\begin{bmatrix}2\\-1\\5\\0\end{bmatrix}}_{\in Null(A)}$

similarly, ② $\Rightarrow$ $\begin{bmatrix}5\\0\\-6\\-1\end{bmatrix} \in Null(A)$

$$\Rightarrow Null(A) = \text{span}\left(\begin{bmatrix}2\\-1\\5\\0\end{bmatrix}, \begin{bmatrix}5\\0\\-6\\-1\end{bmatrix}\right)$$

Linear Equation $Ax = b,\ x \in R^n, b \in R^m$, $A \in \mathbb{R}^{m\times n}$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$Ax \stackrel{?}{=} b$

$col(A) = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \neq \mathbb{R}^3$

$b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

- There exists a solution if

$b \in col(A)$

① $rank(A) \neq 0$

② $b \notin Null(A^T)$

③ $rank(A) \leq \min(n, m)$

- There always exists a solution for any $b \in R^m$ if:

- $\mathbb{R}^m \subseteq col(A)$, $col(A) = span(a_1, \cdots, a_n) = \mathbb{R}^m$

- $rank(A) = m$

- There exists a unique solution if:

- $\begin{cases} b \in col(A) \\ Null(A) = \{0\} \end{cases}$

or $\begin{cases} b \in col(A) \\ \text{columns of } A \text{ indep} \end{cases}$

$\Leftrightarrow rank(A) = n$

suppose

x_1, x_2 are both solutions

$Ax_1 = b$, $Ax_2 = b$

$\Rightarrow A(x_1 - x_2) = 0$

unique $\Rightarrow$ $x_1 = x_2$ always

No nontrivial $x_1 - x_2$ such that $A(x_1 - x_2) = 0$

$\Rightarrow Null(A) = \{0\}$

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