Fall 2021 ME424 Modern Control and Estimation

Lecture Note 2
State Space Models

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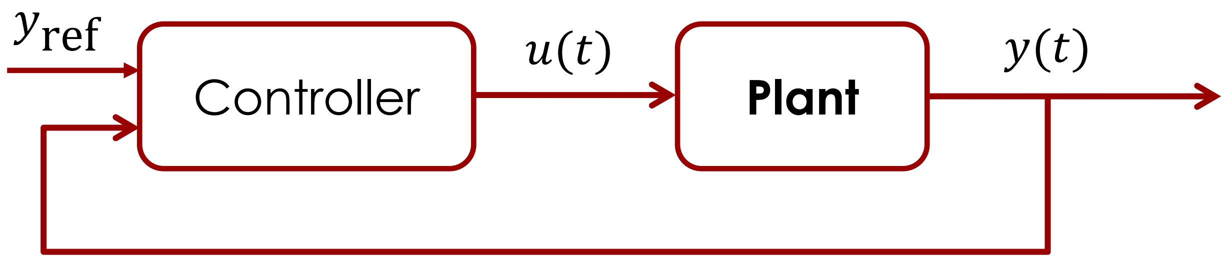
Outline

- **State space model: definition and examples**
- From continuous-time to discrete time model
- From nonlinear to linear model
- State space model $\leftrightarrow$ transfer function

State-space model based feedback control system:

- Goal: determine control input to achieve desired output

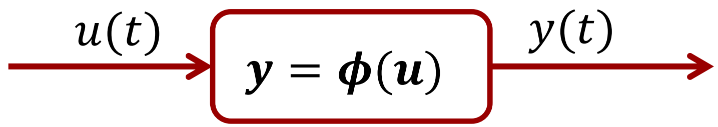


- Controller design is based on plant model
  - Model is different from the actual plant
  - "all models are wrong, but some are useful"

- Modeling approach:
  - First principle
  - Data driven (System ID)

Static vs. Dynamic Systems

- Static system



  - $u(t)$ completely and immediately determines $y(t)$
  - Desired output y_{ref} can be perfectly tracked (in absence of disturbance) by open-loop plant inversion

- **Static vs. Dynamic Systems**
  - **Dynamic system:** differential or difference equation

$$u(t) \longrightarrow \boxed{\mathrm{ODE}(y,u)} \longrightarrow y(t)$$

    - $u(t)$ does not fully determines $y(t)$
    - At time t_0, the output $y(t_0)$ does not fully captures the system "behavior"

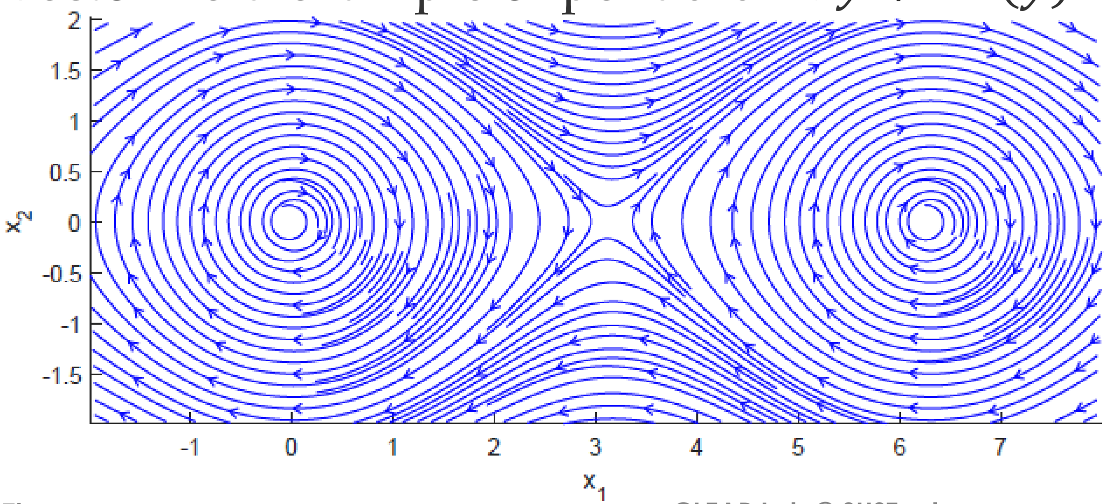
    - **"State": info needed for future evolution,** it separates future from past
    - **State** $x(t_0)$ at time t_0 and **input** $\boldsymbol{u}(\boldsymbol{t})$ over time $[t_o, t_f]$, **completely determines** the system behaviors

- **General continuous-time state space model**

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

  - $x \in R^n$ state vector, $u \in R^m$ control input, $y \in R^p$ output,
  - $f: R^n \times R^m \to R^n$: called **vector field**
  - $h: R^n \times R^m \to R^p$: output function
  - Called autonomous system if there is no control $f(x, u) = f(x)$
  - For autonomous sys, $\hat{x} \in \boldsymbol{R^n}$ is called **equilibrium** if $f(\hat{x}) = 0$

Vector field example of pendulum: $\ddot{y} + \sin(y) = 0$



- **General discrete-time state space model**

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = h(x(k), u(k))$$

  - $x \in R^n$ state vector, $u \in R^m$ control input, $y \in R^p$ output
  - $f: R^n \times R^m \to R^n$: state update equation
  - $h: R^n \times R^m \to R^p$: output function
  - Called autonomous system if there is no control $f(x, u) = f(x)$
  - For autonomous sys, $\hat{x} \in \boldsymbol{R^n}$ is called **equilibrium** if $\hat{x} = f(\hat{x})$

- Discrete-time system:
  - Some discrete-time system is obtained from continuous time model by sampling
  - Some systems naturally evolve in discrete time.

- **Linear Systems:** system is called linear if:

Continuous time

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$$
$$\mathbf{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u},$$

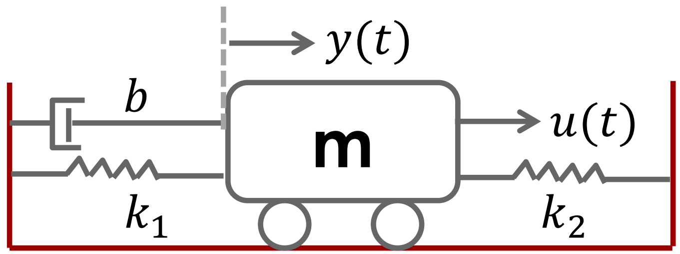
Discrete time

$$\boldsymbol{x}(\boldsymbol{k}+\mathbf{1}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{k}), \boldsymbol{u}(\boldsymbol{k})) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{k}),$$
$$\mathbf{y}(\boldsymbol{k}) = \boldsymbol{h}(\boldsymbol{x}(\boldsymbol{k}), \boldsymbol{u}(\boldsymbol{k})) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{k}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{k}),$$

for some matrices A, B, C, D

- **State-space modeling:**
  - Find the functions $f(\cdot,\cdot), h(\cdot,\cdot)$
  - Or find A, B, C, D matrices if the system is linear

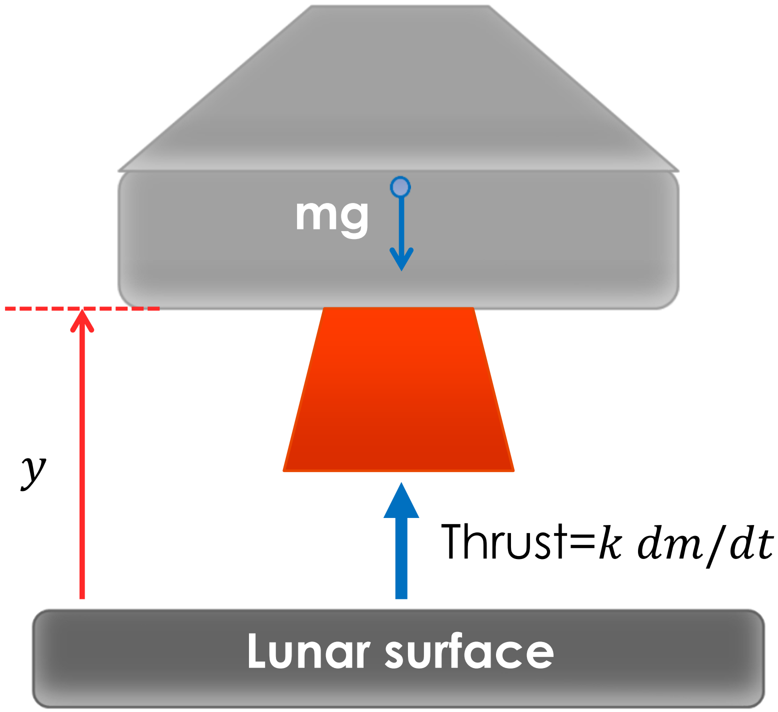
Example 1: Consider spring-damper cart system with zero initial conditions (initially at y = 0 and not moving). No friction



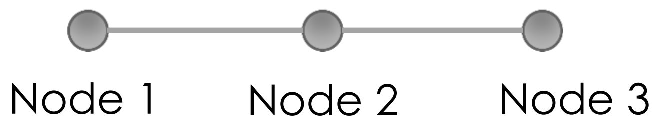
- Differential equation model

- State space model of Example 1 (infinitely many)

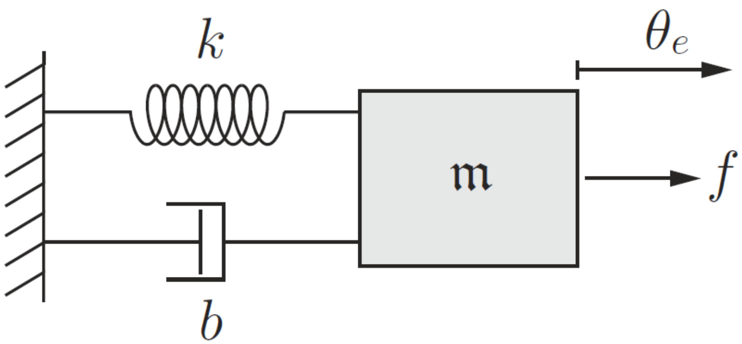
- **Example 2**: soft landing of a lunar module, $u = \frac{dm}{dt}$

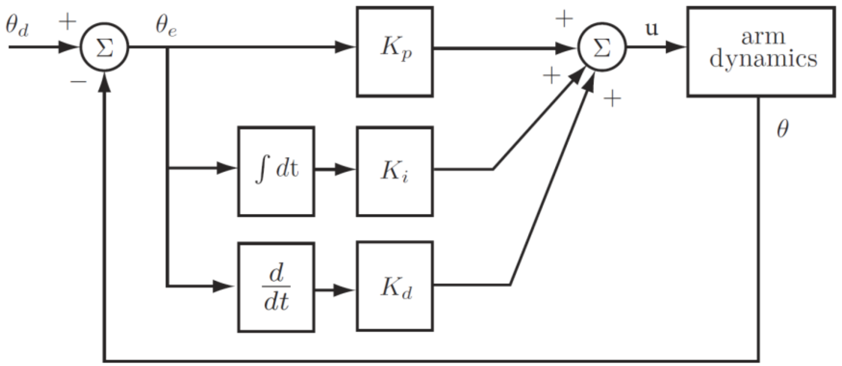


- **Example 3**: Sensor Network
  - Each iteration, exchange measurements with neighbors
  - The updated value is the average of its own value with the neighbors



▪ **Example 4**: PID for spring-damper system





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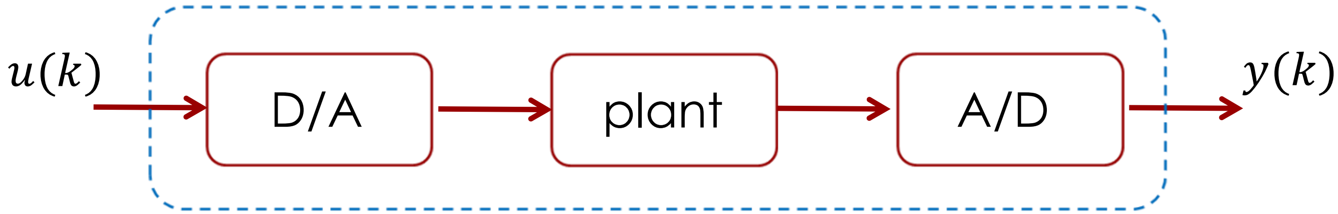
From continuous time to discrete time model



- Approximate differential equation with difference equation
  - Euler forward rule:

From continuous-time to discrete-time model

- General nonlinear case:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

From continuous-time to discrete-time model

- Linear case:

$$\dot{x} = A_c x + B_c u,$$
$$\mathrm{y} = C_c x + D_c u,$$

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From nonlinear to linear

- Given model: $x(k+1) = f(x(k), u(k)),\ y(k) = h(x(k), u(k))$ and operating point: $(\hat{x}, \hat{u})$
- Goal: find a linearized model around $(\hat{x}, \hat{u})$

- Jacobian matrix of multivariable function $f: R^n \rightarrow R^m$

- Example of Jacobian matrix: $f(z) = \begin{bmatrix} 2z_1 + e^{z_2} \\ \log(z_3) + \frac{1}{z_2} \end{bmatrix}$, $\hat{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- Taylor expansion of multivariate function
  - General expression: $f(z) = f(\hat{z}) + \left(\frac{\partial f}{\partial z}(z) \Big|_{z=\hat{z}} \right) \Delta z$+H.O.T

- Linearization around $(\hat{x}, \hat{u})$ using Taylor expansion:

$$f(x,u) \approx f(\hat{x},\hat{u}) + \overbrace{\left(\frac{\partial f(x,u)}{\partial x} \bigg|_{x=\hat{x},u=\hat{u}} \right)}^{\hat{A}} \cdot \overbrace{(x - \hat{x})}^{\Delta x} + \overbrace{\left(\frac{\partial f(x,u)}{\partial u} \bigg|_{x=\hat{x},u=\hat{u}} \right)}^{\hat{B}} \cdot \overbrace{(u - \hat{u})}^{\Delta u}$$

$$= \hat{A} \cdot \Delta x + \hat{B} \cdot \Delta u + f(\hat{x},\hat{u})$$

$$h(x,u) \approx h(\hat{x},\hat{u}) + \underbrace{\left(\frac{\partial h(x,u)}{\partial x}\bigg|_{x=\hat{x},u=\hat{u}}\right)}_{\hat{C}} \cdot \underbrace{(x-\hat{x})}_{\Delta x} + \underbrace{\left(\frac{\partial h(x,u)}{\partial u}\bigg|_{x=\hat{x},u=\hat{u}}\right)}_{\widehat{D}} \cdot \underbrace{(u-\hat{u})}_{\Delta u}$$

$$\Delta y \coloneqq y - h(\hat{x},\hat{u}) \approx \hat{C} \cdot \Delta x + \widehat{D} \cdot \Delta u$$

- **Example:** $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \sin(x_2(k)) + \cos(u_2(k)) \\ x_1(k)x_2(k) + u_1u_2(k) \end{bmatrix}$$

$$y(k) = \cos(x_2(k)) + 2x_1(k) \qquad \hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{u} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}$$

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- **From state space to transfer function**

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

  - Let $X(z), U(Z), Y(z)$ be the z-transforms of $x(k), u(k), y(k)$
  - z-transform: $X(z) \triangleq \sum_{k=0}^{\infty} x(k) z^{-k}$

- **From state space to transfer function**

$$\mathrm{x}(\mathrm{k}+1) = Ax(k) + Bu(k)$$
$$\mathrm{y}(\mathrm{k}) = Cx(k) + Du(k)$$

  - Recall: if $x(k) \leftrightarrow X(z)$, then $x(k+1) \leftrightarrow zX(z) - zx(0)$

- **From transfer function to state space model**
  - **Realization problem**: given transfer function $H(z)$, find (A, B, C, D)

- Single-input-single-output system:

$$H(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

- Procedure:
  - First write the transfer function in the above canonical form
  - One possible realization is:

- **Example**: $y(k+1)+3y(k-2)=2u(k-1)$
  - First find transfer function: