Fall 2021 ME424 Modern Control and Estimation

Lecture Note 4
Stability, Controllability, and Observability

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- **So far, we have learned:**
  - What is a state space model
  - How to derive state space model from physical systems, nonlinear model, continuous time model, and transfer function model
  - How to identify state space model from input-out data pairs
- **Question: how to use a state-space model?**
  - Predict system output: solution to a state space model
  - Analysis of behavior: stability/controllability/observability
  - Design controller
- **The goal of this lecture note: Given a state space model**
  - Derive its solution analytically
  - Stability
  - Controllability
  - Observability

Outline

- **State Space Solutions**
- Internal Stability
- Controllability
- Observability
- Invariance under Similarity Transformation

- General state space model:

$$x(k+1) = A(k)x(k) + B(k)u(k),$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

  - If system matrices $\big(A(k), B(k), C(k), D(k)\big)$ change over time k, then system is called **Linear Time Varying (LTV)** system

  - If system matrices are constant w.r.t. to time, then the system is called a **Linear Time Invariant (LTI)** System

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

- Derivation of Solution to LTI state space system:

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

  - given initial state $x(0) = \hat{x}$, and control sequence $u(0), \ldots, u(k)$, $k \geq 0$, we have $\boldsymbol{x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)}$

- A large portion of control applications can be transformed into a ***regulation* problem**
  - Regulation problem: keep certain function of the state $x(k)$ or output $y(k)$ close to a known constant reference value under disturbances and model uncertainties

For example:

- Keep inverted pendulum at upright position ($\theta = 0$)
- Maintain a desired attitude of spacecraft or aircraft
- Air conditioner regulate temperate close to setpoint (e.g. 75F)
- Cruise control maintain a constant speed despite uncertain road conditions
- Converter maintains a desired voltage level for different loads

  - If reference $y_{\text{ref}}(t)$ is changing, this is no longer a regulation problem (becomes a **tracking problem**, which will be discussed later in this class)

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- **Internal Stability** (with $u(k) \equiv 0$, i.e. concerned with zero-input state response)

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

  - Asymptotic stable: $||x(k)|| \to 0$, as $k \to \infty$, for all initial state $\hat{x}$

  - Marginal stable: $||x(k)|| \le M,$ for all $k = 1,2, \ldots$

- Recall state space solution for linear systems:

$$\boldsymbol{x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)}$$

- Therefore, or linear system, the key for stability analysis is to understand how A^k behave as $k \to \infty$

- **Case 1:** diagonal matrix: e.g. $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

- **Case 2:** diagonalizable matrix, i.e. $\exists\, T$ such that $A = TDT^{-1}$

- **Case 3:** Unfortunately, ***not all*** square matrices are diagonalizable
  - e.g.: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is **not diagonalizable**

- **Theorem (Internal stability)**: LTI $(\boldsymbol{A}, \boldsymbol{B})$ is **asymptotically stable** if all eigs of A satisfies $|\boldsymbol{\lambda_i}| < \mathbf{1}$

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Nontrivial Fact (by Cayley Hamilton theorem): For a square matrix $A \in R^{n\times n}$, A^k for arbitrary k can be written as a linear combination of $\{I, A, A^2, \dots, A^{n-1}\}$

$$A^k = \alpha_k A^n + \alpha_{k-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I$$

- E.g.: $A = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$, $A^{10} = \begin{bmatrix} 365 & 1159 \\ 1159 & 3842 \end{bmatrix}$

We can write $A^{10} = \alpha_0 I + \alpha_1 A$

- ***k*-step reachability:**
  - Given a system (A, B), a final state x_f is called k-step reachable from an initial state x_0 if there exists an input sequence $u(0), \dots, u(k-1)$ such that $x(k) = x_f$
  - $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)$

    $= A^k x_0 + Bu(k-1) + ABu(k-2) + \cdots + A^{k-1} B u(0)$
    - Matrix form:
  - **Reachability Lemma:** a final state x_f is k-step reachable from x_0 if

$$x_f - A^k x_0 \in range\left([B, AB, \dots, A^{k-1}B]\right)$$

- Reachability Examples:
  - $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find out what state is reachable in 1 step, 2 steps, 10 steps?
  - What if $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$?

- **Controllability**
  - A system matrix pair (A, B) is called **controllable** if any state $x_f \in R^n$ is reachable from any initial state $x_0 \in R^n$ in **finite** time steps
    - In other words, for any initial state $x_0 \in R^n$ and final sate $x_f \in R^n$, we can find a control input $u(\cdot)$ to steer the system from x_0 to x_f in finite time steps
  - According to Reachability lemma, this requires:

$$x_f - A^k x_0 \in range\left([B, AB, \dots, A^{k-1}B]\right), \forall x_f, x_0$$

  - So system is controllable if and only if

$$rank\left([B, AB, \dots, A^{k-1}B]\right) = n \text{ for some finite } k$$

  - One way to check controllability is to keep increasing k to see whether $[B, AB, \dots, A^{k-1}B]$ is n or not
  - Cayley Hamilton Theorem indicates that there is no need to check the case for $k > n$

- **Controllability Test**: (A, B) is controllable if and only if the controllability matrix $\boldsymbol{M_c} \triangleq [\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A^{n-1}B}]$, is full rank
  - Cayley-Hamilton theorem implies: $rank([\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A^{k-1}B}]) = rank([\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A^{n-1}B}])$, for all $k \geq n$

- Examples:
  - $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
  - $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

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Observability:

- A system (A, B, C) is called **observable** if any initial state $x_0 \in R^n$ can be uniquely determined given the system input trajectory $u(0), u(1), \dots, u(k-1)$ and output trajectory $y(0), y(1), \dots, y(k)$ for some finite k

- What is the relation between output and initial state?
  - Note $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)$

    $= A^k x_0 + Bu(k-1) + ABu(k-2) + \cdots + A^{k-1} Bu(0)$

- Continue derivation for relation between $y(\cdot)$ and x_0

- In vector form, we have: $Y_k = O_k x_0 + T_k U_k \Rightarrow O_k x_0 = Y_k - T_k U_k$
  - O_k maps initial state x_0 to ouput over time $[0, k-1]$
  - T_k maps input to output over time $[0, k-1]$

- $Null(O_k)$ gives ambiguity in determining x_0

- x_0 can be uniquely determined if $Null(O_k) = \{0\}$, i.e. $rank(O_k) = n$, for some finite k

- Input u does not affect **ability** to determine x_0
  - its effect can be subtracted out
  - So we often say system (A, C) is observable or not (No need to mention B)

- Observability Test: (A,C) is observable if and only if observability matrix M_o is full rank, where $M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$
  - Note: (A,C) observable $\Leftrightarrow$ Any x_0 can be uniquely determined by $u(0), \dots, u(k-1), y(0), \dots, y(k-1)$, for a finite k

- Observability example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, y(k) \equiv 2, u(k) \equiv 0, k = 0,1,2$$

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- Change of basis vectors for equivalent system representation
  - Canonical coordinate system:
  - New coordinate system: basis vectors
  - Relation: $x = Pz$

- Dynamics in new coordinate systems:
  - $x(k) = Pz(k) \Rightarrow z(k) = P^{-1}x(k)$
  - Dynamics in original coordinate:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k),$$

  - Dynamics in the new coordinate

$$z(k+1) = \hat{A}z(k) + \hat{B}u(k), \; y(k) = \hat{C}z(k) + \hat{D}u(k)$$

  - Derive $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$

  - System $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is called **equivalent** to original system (A, B, C, D)

- Similarity transformation does not change system stability, controllability and observability
  - A and PAP^{-1} have the same Jordan form, so similarity transformation does not change stability, i.e., **A is (asymptotically) stable iff $\hat{A}$ is (asymptotically) stable**

- $\widehat{M}_c = P^{-1}M_c$, with this it can be shown that M_c full rank iff $\widehat{M}_c$ full rank

therefore, **$(\boldsymbol{A}, \boldsymbol{B})$ controllable iff $(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{B}})$ controllable**

$\widehat{M}_o = M_o P$, similarly, this implies **$(\mathbf{A}, \mathbf{C})$ observable iff $(\widehat{\mathbf{A}}, \widehat{\mathbf{C}})$ observable**