Fall 2021 ME424 Modern Control and Estimation

Linear Algebra Review: Part III
Geometry ~ linear algebra

Prof. Wei Zhang
Department of Mechanical and Energy Engineering
SUSTech Institute of Robotics
Southern University of Science and Technology

zhangw3@sustech.edu.cn
https://www.wzhanglab.site/

Outline

- Inner product
- Projection
- Geometric sets
  - Lines
  - Convex set and cone
  - Hyperplanes
  - Polytopes
  - Ball, ellipsoid

Inner Product

- Inner product of vectors in R^n: $< v, w > =$

- Norm of $v \in R^n$

- Angle between $v, w \in R^n$

- Orthogonality:

Inner Product

- General inner product $<\cdot,\cdot>: V \times V \rightarrow R$
  - maps each pair in a vector space to a scaler
  - satisfies several key properties: linearity, conjugate, positive definiteness …

- Inner product of matrices in $R^{m\times n}$: $< A, B > =$

- Inner product of two functions f, g on interval $[a, b]$:

Projection

- Projection of $v \in R^n$ along direction e

- $\{e_1, \dots, e_k\}$ be orthonormal basis of vector space V, then any $v \in V$,

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k$$

Projection

- **Fourier series**: Consider a vector space of periodic functions:
  $$V = \{\text{integrable functions over } [0,2\pi)\}$$

- Inner product: $< f, g > = \int_0^{2\pi} f(x)g(x)dx$

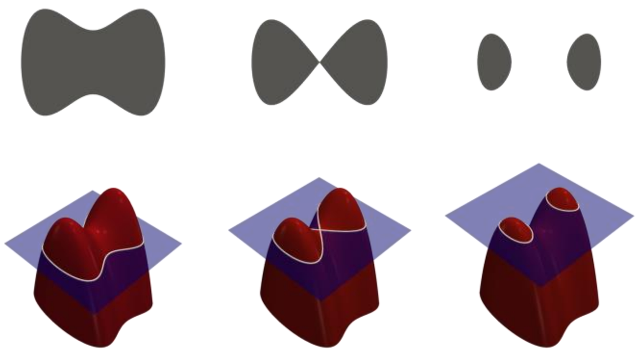
- Basis: $B = \{1,\ \cos x, \sin(x), \cos(2x),\ \sin(2x), \dots\}$

- $f \in V$, then

Representation of Geometric Objects / sets

- Implicit representation via (sub)-level sets:

$$\{x \in R^n:\ f(x) = 0\} \quad \text{or} \quad \{x \in R^n: f(x) \leq 0\}$$



- Explicit representation: $\{x(\alpha) \in R^n:\ \alpha \text{ satisfies certain conditions}\}$

Some Simple Geometric Sets

- Line segment: Given $x_1 \neq x_2 \in R^n$: $\{x_2 + \alpha\,(x_1 - x_2): \ \alpha \in [0,1]\}$

- Line (explicit): $\{x_2 + \alpha\,(x_1 - x_2): \ \alpha \in R\}$

- Line: (implicit representation)
  - e.g. in R^2: $\{x \in R^2: \ a^T x = b\}$

Some Simple Geometric Sets

- Hyperplanes (Implicit): $\{x \in R^n : a^T x = b\}$

- Hyperplanes (Explicit): $\{x \in R^n : a^T x = b\} \Rightarrow \{x_0 + \sum_i \alpha_i v_i : \ \alpha_i \in R, i = 1, \dots, n-1\}$

- Halfspaces: $\{x \in R^n : a^T x \leq b\}$

Some Simple Geometric Sets

- Convex set: A set S is called convex if

$$x_1, x_2 \in S \Rightarrow \alpha x_1 + (1-\alpha) x_2 \in S$$

- **Convex combination** of $x_1, \ldots, x_k \in R^n$

$$\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k : \alpha_i \geq 0, \sum \alpha_i = 1\}$$

- Convex hull $\overline{co}(S)$: set of all convex combinations of points in S

Some Simple Geometric Sets

- **Cone**: A set S is called a cone if $x \in S \Rightarrow \lambda x \in S, \ \forall \lambda \geq 0$

- **Conic combination** of $x_1, \dots, x_k \in R^n$

$$cone(x_1, \dots, x_k) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k : \alpha_i \geq 0\}$$

Some Simple Geometric Sets

- **(Convex) Polyhedron**: intersection of a finite number of half spaces
$$P = \{x : Ax \le b\}$$

- **Polyhedral cone**: intersection of finitely many halfspaces that contain the origin:
$$P = \{x : Ax \le 0\}$$

- **Polytope**: bounded polyhedron

Some Simple Geometric Sets

- Polyhedron (vertex representation):

$$P = \overline{co}(v_1, \ldots, v_m) \oplus cone(r_1, \ldots, r_q)$$

Some Simple Geometric Sets

- Euclidean balls: $B(x_c, r) = \{x \in R^n : \|x - x_c\|_2 \leq r\}$ or $B(x_c, r) = \{x_c + ru : u \in R^2, \|u\|_2 \leq 1\}$

- Ellipsoids: $E = \{x \in R^n : (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$ or $E = \{x_c + Au : u \in R^2, \|u\|_2 \leq 1\}$