Fall 2021 ME424 Modern Control and Estimation

Lecture Note 4
Stability, Controllability, and Observability

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- **So far, we have learned:**
  - What is a state space model
  - How to derive state space model from physical systems, nonlinear model, continuous time model, and transfer function model
  - How to identify state space model from input-out data pairs
- **Question: how to use a state-space model?**
  - Predict system output: solution to a state space model
  - Analysis of behavior: stability/controllability/observability
  - Design controller
- **The goal of this lecture note: Given a state space model**
  - Derive its solution analytically
  - Stability
  - Controllability
  - Observability

Outline

- **State Space Solutions**
- Internal Stability
- Controllability
- Observability
- Invariance under Similarity Transformation

LTI if M does not chang with time

- General state space model:

$$x(k+1) = A(k)x(k) + B(k)u(k),$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

  - If system matrices $\big(A(k), B(k), C(k), D(k)\big)$ change over time k, then system is called **Linear Time Varying (LTV)** system

  - If system matrices are constant w.r.t. to time, then the system is called a **Linear Time Invariant (LTI)** System

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k) \end{cases}$$

- Derivation of Solution to LTI state space system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k) \end{cases}$$

- given initial state $x(0) = \hat{x}$, and control sequence $u(0), \dots, u(k)$, $k \geq 0$, we have $\boxed{\boldsymbol{x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j)}}$

$$x(1) = Ax(0) + Bu(0) = A\hat{x} + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A^2\hat{x} + ABu(0) + Bu(1)$$

$$x(3) = Ax(2) + Bu(2) = \underbrace{A^3\hat{x}}_{\text{depends on IC}} + \underbrace{A^2Bu(0) + ABu(1) + Bu(2)}_{\text{depend on input history}}$$

For arbitrary k:

$$x(k) = A^k\hat{x} + A^{k-1}Bu(0) + A^{k-2}Bu(1) + \cdots + ABu(k-2) + Bu(k-1)$$

$$= \boxed{A^k\hat{x} + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j)} \quad \text{state trajectory}$$

$$\boxed{y(k) = \underbrace{CA^k\hat{x}}_{\text{zero-input response}} + \underbrace{\sum_{j=0}^{k-1} CA^{k-1-j} Bu(j) + Du(k)}_{\text{zero-state response}}} \quad \text{output trajectory}$$

- A large portion of control applications can be transformed into a ***regulation* problem**

  - Regulation problem: keep certain function of the state $x(k)$ or output $y(k)$ close to a known constant reference value under disturbances and model uncertainties

For example:

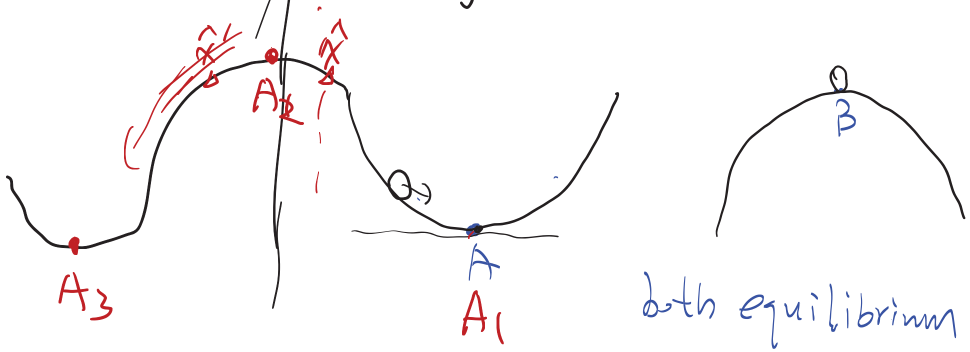
- Keep inverted pendulum at upright position ($\theta = 0$)
- Maintain a desired attitude of spacecraft or aircraft
- Air conditioner regulate temperate close to setpoint (e.g. 75F)
- Cruise control maintain a constant speed despite uncertain road conditions
- Converter maintains a desired voltage level for different loads

  - If reference $y_{ref}(t)$ is changing, this is no longer a regulation problem (becomes a **tracking problem**, which will be discussed later in this class)

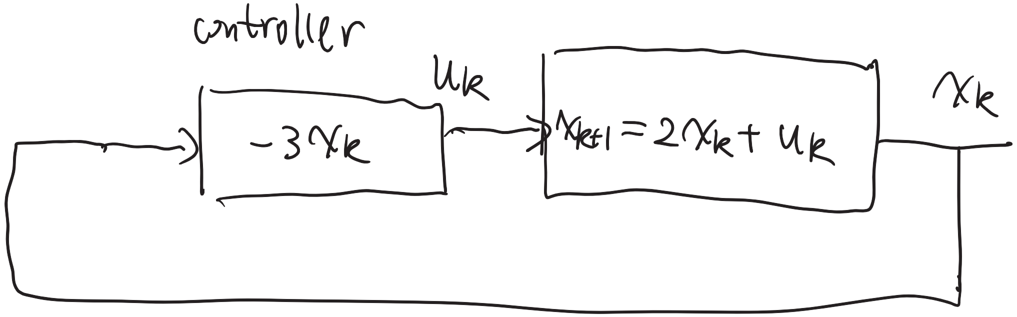
Outline

- State Space Solutions
- **Internal Stability** BIBO stability. with respect to equilibrium

$$x_{k+1} = f(x_k) = x_k \qquad \dot{x} = f(x) = 0$$



- Controllability
- Observability
- Invariance under Similarity Transformation



$$\Longleftrightarrow \quad x_{k+1} = -x_k$$

BIBO: bounded input bounded output stability (external stability)

- **Internal Stability** (with $u(k) \equiv 0$, i.e. concerned with zero-input state response)

$$\left\{\begin{array}{l}\text{is mainly about autonomous system without control} \\ \text{or about the closed-loop system for which the controller has already been incorporated}\end{array}\right.$$

$$x(k+1) = Ax(k) + Bu(k),$$
$$y(k) = Cx(k) + Du(k)$$

  - $\boxed{\text{Asymptotic stable:}}$ $||x(k)|| \to 0$, as $k \to \infty$, for all initial state $\hat{x}$

e.g. $x(k) = \begin{bmatrix} 2\left(\frac{1}{2}\right)^k \hat{x}_1 \\ \left(\frac{1}{3}\right)^k \hat{x}_2 \end{bmatrix}$, $\|x(k)\| \to 0$, for all $\hat{x}$

  - Marginal stable: $||x(k)|| \leq M$, for all $k = 1,2, \ldots$

$x(k) = \begin{bmatrix} \sin\left(\frac{\pi}{2}k\right) \hat{x}_2 \\ 2\hat{x}_1 + \left(\frac{1}{2}\right)^k \hat{x}_2 \end{bmatrix}$

e.g.: $x(k+1) = 2x(k) + u(k)$

Control law: $u(k) = -3x(k)$

$\boxed{x(k+1) = -x(k)}$ closed-loop

- Recall state space solution for linear systems:

$$\boxed{\boldsymbol{x(k) = A^k \hat{x}}} + \cancel{\sum_{j=0}^{k-1} \boldsymbol{A^{k-j-1} B u(j)}}$$

For internal stability, let $u(k) \equiv 0$

- Therefore, or linear system, the key for stability analysis is to understand how A^k behave as $k \to \infty$

$A \in \mathbb{R}^{n \times n} \Rightarrow n$ eigs $\qquad \det(\lambda I - A) = 0$

- **Case 1:** diagonal matrix: e.g. $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \lambda_3^k \end{bmatrix}$$

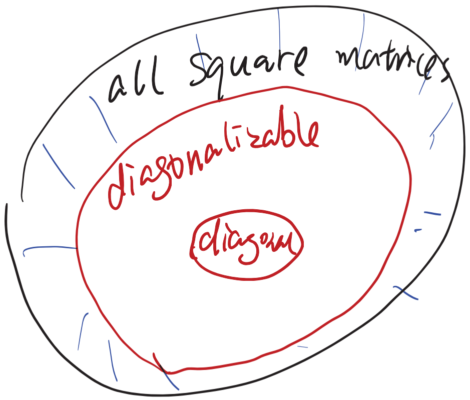
For internal stability, $u = 0$, $x(k) = A^k \hat{x}$, so the system is asym

iff $|\lambda_1| < 1$, $|\lambda_2| < 1$, $\cdot$, $|\lambda_3| < 1$, i.e. all eigs lie inside unit circle

- **Case 2:** diagonalizable matrix, i.e. $\exists\, T$ such that $A = TDT^{-1}$

D is diagonal.

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$



In this case, $A^k = TD\cancel{T^{-1}T}DT^{-1} \cdots\cdots (TDT^{-1})$

$$= TD^kT^{-1} = T \underbrace{\begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \lambda_3^k \end{bmatrix}} T^{-1}$$

- system is asym stable iff all eigs lie inside unit circle

- **Case 3:** Unfortunately, ***not all*** square matrices are diagonalizable
  - e.g.: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is **not diagonalizable**

Recall: All square matrices are block diagonalizable. they can be put into "Jordan" form

$\boxed{\text{But conclusion is the same}}$

In all cases

- **Theorem (Internal stability)**: LTI $(\boldsymbol{A}, \boldsymbol{B})$ is **asymptotically stable** if all eigs of A satisfies $|\boldsymbol{\lambda_i}| < \mathbf{1}$

* strictly inside unit circle

- More about stability
  - Asym stable : $x(k) = A^k x_0 \longrightarrow 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$
    (assumed $u(k) \equiv 0$)
  - stability Test: eig(A) $\in$ Unit Circle $|eig(A)| < 1$

- Most general case: A can always be written as $\underline{A = TJT^{-1}}$
  where J is Jordan form of A
  - when A is diagonalizable, J = diagonal matrix
  - ... A is not ......, $J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & J_3 & \\ & & & \ddots \end{bmatrix}$ → Jordan blocks
  - $x(k) = A^k x_0 = (TJ\cancel{T^{-1}}\,\cancel{T}JT^{-1} \cdots\cdots)x_0 = TJ^kT^{-1} \cdot x_0$

$$= T \cdot \begin{bmatrix} J_1^k & & & 0 \\ & J_2^k & & \\ & & J_3^k & \\ 0 & & & \ddots \end{bmatrix} T^{-1} \cdot x_0$$

we need all Jordan blocks $\to 0$ as $k \uparrow \infty$

e.g. $$J = \begin{bmatrix} -1 & & & & & & \\ & 2 & 1 & & & & \\ & 0 & 2 & & & & \\ & & & 4 & 1 & 0 & \\ & & & 0 & 4 & 1 & \\ & & & 0 & 0 & 4 & \\ & & & & & & -1 \end{bmatrix}$$

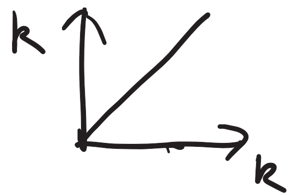
J_1, J_2, J_3, J_4

e.g. 2×2 block $J_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$, $J_i^2 = \begin{bmatrix} \lambda_i^2 & 2\lambda_i \\ 0 & \lambda_i^2 \end{bmatrix}$, $J_i^3 = \begin{bmatrix} \lambda_i^3 & 3\lambda_i^2 \\ 0 & \lambda_i^3 \end{bmatrix}$

$$J_i^k = \begin{bmatrix} \lambda_i^k & k \cdot \lambda_i^{k-1} \\ 0 & \lambda_i^k \end{bmatrix},$$

$J_i^k \to 0$ iff $|\lambda_i| < 1$

$k \cdot \lambda_i^k$



3x3 block: $J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \xRightarrow{\text{V.F.T.}} J_i^k = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \frac{k(k-1)}{2!}\lambda_i^{k-2} \\ 0 & \lambda_i^k & k\lambda_i^{k-1} \\ 0 & 0 & \lambda_i^k \end{bmatrix}$

- $\bigcirc$ is derivative of $\triangle$
- $\square$ is derivative of $\bigcirc / 2!$
- $J_i \to 0$ iff $|\lambda_i| < 1$

- Claim: Asym stability $\iff$ $|\text{eig}(A)| < 1$

- stability in classical control. Real_poles < 0

· Recall $\rightarrow \boxed{H(s)} \rightarrow$ is stable iff poles$(H(s)) \in$ OLHP

$z = e^{sT}$

discrete time $\rightarrow \boxed{H(z)} \rightarrow$ is stable iff poles$(H(z)) \in$ Unit circle

· These transfer func based stability is called external stability or BIBO.

· Bounded input bounded output (BIBO): [bounded input must produce bounded output]

· bounded signal $u(k)$. $|u(k)| < M$ for $k \in \mathbb{N}$

• It's fine for output to be unbounded if input is unbounded

- System in state space form: A, B, C, D.

. . . . transfer func. : $H(z) = \underline{C(zI-A)^{-1}B+D}$

what's the relationship between internal and BIBO stability

thm: <1> $\underbrace{\text{if sys } (A,B,C,D) \text{ asym stable}}_{\text{i.e. } |\text{eig}(A)|<1} \Rightarrow \underbrace{\text{it's also BIBO}}_{|\text{poles}(H(z))|<1}$

proof: suppose asym stable $\Rightarrow |\text{eigs}(A)|<1$

$H(z) = C(zI-A)^{-1}B+D$, where $(zI-A)^{-1} = \dfrac{[\text{co-factor of } A]^T}{\det(zI-A)}$

Note: $\det(zI - A) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$

characteristic polynomial of z, $\det(zI-A)=0$ has n roots that are eigs of A

$\lambda_1, \lambda_2, \ldots \lambda_n$

$\Rightarrow$ All poles of $H(z)$ must be eigs of A $\Rightarrow$ $|\text{poles}(H(z))| < 1$

(2) BIBO $\not\Rightarrow$ Asym stability

Some eigs of A may not show up in the expression of $H(z)$

(Maybe cancelled by co-factor of B, C)

e.g.: $A = \begin{bmatrix} 0.5 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $C = [1 \quad 0]$

- Is it asym stable? <u>No</u> $\text{eig}(A) = \{0.5, 3\}$

- Is it BIBO?

$$H(z) = C(zI - A)^{-1}B$$

$$(zI - A)^{-1} = \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0.5 & 1 \\ 0 & 3 \end{bmatrix} \right)^{-1} = \begin{pmatrix} z-0.5 & -1 \\ 0 & z-3 \end{pmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{z-0.5} & \frac{1}{(z-0.5)(z-3)} \\ 0 & \frac{1}{z-3} \end{bmatrix}$$

$$H(z) = [1 \;\; 0] \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{z-0.5}$$

$$\text{pole}(H(z)) = 0.5 < 1$$

$\Rightarrow$ BIBO

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- State Space Solutions
- Internal Stability
- **Controllability**
- Observability
- Invariance under Similarity Transformation

Nontrivial Fact (by Cayley Hamilton theorem): For a square matrix $A \in R^{n\times n}$, A^k for arbitrary k can be written as a linear combination of $\{I, A, A^2, \dots, A^{n-1}\}$

$$A^k = \cancel{\alpha_k A^n + \alpha_{k-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I}$$

$$= \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

- E.g.: $A = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$, $A^{10} = \begin{bmatrix} 365 & 1159 \\ 1159 & 3842 \end{bmatrix}$

$n = 2$

We can write $A^{10} = \alpha_0 I + \alpha_1 A = \begin{bmatrix} \alpha_0 + \alpha_1 & -\alpha_1 \\ -\alpha_1 & \alpha_0 - 2\alpha_1 \end{bmatrix} = \begin{bmatrix} 365 & 1159 \\ 1159 & 3842 \end{bmatrix}$

$\Rightarrow \alpha_1 = -1159,\ \alpha_0 = 1524 \Longrightarrow$ V.F.Y.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $A^{100} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

- ***k*-step reachability:** $x_0 \xrightarrow{k\text{-step}} x_f$
  - Given a system (A, B), a final state x_f is called k-step reachable from an initial state x_0 if there exists an input sequence $u(0), \dots, u(k-1)$ such that $x(k) = x_f$

    @ $u(k) \in \mathbb{R}^m$

  - $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)$

    $= A^k x_0 + Bu(k-1) + ABu(k-2) + \cdots + A^{k-1}Bu(0)$
    - Matrix form: $x(k) - A^k x_0 = \underbrace{[B \;\vdots\; AB \;\vdots\; A^2B \;\cdots\; A^{k-1}B]}_{k\cdot m \text{ columns},\ n \text{ rows}} \underbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}}_{k\cdot m \text{ rows in total}}$

      m columns, m-columns; m-rows, m-rows

      $\triangleq M_k$, $\triangleq U_k$

We want to find $u(0), \dots, u(k-1)$ so that $x(k) = x_f$. In other words, we want solve

$$\underbrace{(x_f - A^k x_0)}_{n\times 1} = \underbrace{M_k}_{n\times(k\cdot m)} \cdot \underbrace{U_k}_{(k\cdot m)\times 1}$$

unknowns

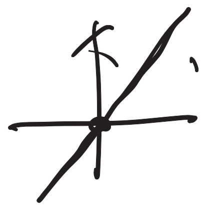
sys has solution $\iff x_f - A^k x_0 \in col(M_k)$

- **Reachability Lemma:** a final state x_f is k-step reachable from x_0 if

$$x_f - A^k x_0 \in range\left([B, AB, \dots, A^{k-1}B]\right)$$

$col(M_k)$

- Reachability Examples:



  - $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find out what state is reachable in 1 step, 2 steps, 10 steps?

in one-step

1-step: $x(1) = Ax_0 + Bu(0) = Bu(0)$, $x_f = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$, for some α

$x_f \in col(B)$

2-step: $x(2) - \cancel{A^2 x_0} = [B \quad AB] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \Rightarrow x(2) \in col\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right)$

$= col\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

⋮

10-step reachable state: $x_f \in col\left(\left[B \quad AB \quad \cdots \quad A^9 B\right]\right) = col\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

  - What if $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$?

1-step reachable state? : $x_f \in col([B]) = \mathbb{R}^2$, i.e. all states in $\mathbb{R}^2$ is 1-step reachable

e.g. pick $x_f = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \Rightarrow u(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

- **Controllability**
  - A system matrix pair (A, B) is called **controllable** if any state $x_f \in R^n$ is reachable from any initial state $x_0 \in R^n$ in **finite** time steps
    - In other words, for any initial state $x_0 \in R^n$ and final sate $x_f \in R^n$, we can find a control input $u(\cdot)$ to steer the system from x_0 to x_f in finite time steps
  - According to Reachability lemma, this requires:

$$\underbrace{x_f - A^k x_0}_{\text{arbitrary} \in \mathbb{R}^n} \in range\left([B, AB, \dots, A^{k-1}B]\right), \forall x_f, x_0$$

$col(M_k) = \mathbb{R}^n$

  - So system is controllable if and only if

$$rank\left([B, AB, \dots, A^{k-1}B]\right) = n \text{ for some finite } k$$

  - One way to check controllability is to keep increasing k to see whether $[B, AB, \dots, A^{k-1}B]$ is n or not
  - Cayley Hamilton Theorem indicates that there is no need to check the case for $k > n$

- **Controllability Test**: (A, B) is controllable if and only if the controllability matrix $M_c \triangleq [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$, is full rank $\Leftrightarrow \text{rank}(M_c) = n$

$A \to \mathbb{R}^{n\times n}$, $B \to \mathbb{R}^{n\times m}$

  - Cayley-Hamilton theorem implies: $rank([B \;\; AB \;\; \cdots \;\; A^{k-1}B]) = rank([B \;\; AB \;\; \cdots \;\; A^{n-1}B])$, for all $k \geq n$

- controllable $\Leftrightarrow$ $\text{rank}([B \;\; AB \cdots A^{k-1}B]) = n$, for some finite k

- By Cayley-Hamilton Thm $\Rightarrow$ $\text{rank}(\underbrace{[B \;\; AB \cdots A^{k-1}B]}_{M_k}) = \text{rank}(\underbrace{[B \;\; AB \cdots A^{n-1}B]}_{M_c})$ for all $k \geq n$

why? : suppose $k = n+1$, $M_k = [B \;\; AB \cdots A^{n-1}B \;\; A^nB]$ $\quad B = [b_1 \;\; b_2 \cdots b_m]$

we know $A^n = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1}$ $\Rightarrow \underbrace{A^nB}_{n\times m} = [A^nb_1 \;\; A^nb_2 \;\; \cdots \;\; A^nb_m]$

$\Rightarrow A^nB$ is a linear combination of columns of M_c

$\Rightarrow \text{rank}(M_k) = \text{rank}(M_c)$

- Examples:

e.g.: $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_f = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, check $\underline{x_k - A^k x_0} = [B \; AB \; \cdots A^{k-1} B] \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$

1-step: $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underline{u_0}$ choose $u_0 = 3 \Rightarrow x_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = x_f$

But if we pick $x_f = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

  - $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$n=2$, $m=1$, $M_c = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $\text{rank}(M_c) = 1 < 2 \Rightarrow$ uncontrollable

then we can $\underline{\text{never}}$ reach x_f for any k

  - $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

- $n=2$, $m=2$. $M_c = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $\text{rank}(M_c) = 2 \Rightarrow$ controllable

$= n$

  - $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$n=3$, $m=1$

$M_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}$, $\text{rank}(M_c) = 3 = n \Rightarrow$ controllable

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$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

– Motivating discussion for observability

- To design state-feedback control, we need to know the "state" x
- but often times we can only measure output : $y(k) = Cx(k) + Du(k)$

$C \in \mathbb{R}^{p \times n}$

1° if $\underbrace{C \text{ is square and invertible}}_{\text{equivalent to directly measure all the state}}$, then $x(k) = C^{-1}(y(k) - Du(k))$

2° "C" is not invertible :

$$C = \begin{bmatrix} \text{———} \\ \text{———} \end{bmatrix}$$

$$y(0) = Cx(0) + Du(0)$$
$$y(1) = Cx(1) + Du(1)$$
$$\vdots \qquad \vdots$$

Question: Can we use $\{u(j), y(j)\}_{j=0}^{k}$ to estimate $x(0), x(1), \dots, x(k)$?

Note: All $x(k)$ depends on $x(0)$, and $\underbrace{u(0), \dots u(k-1)}_{\text{known}}$

definition of observability

$\Longleftrightarrow$ Question : $\boxed{\text{Can we use } \{u(j), y(j)\}_{j=0}^{k} \text{ to uniquely determine } x_0}$

Observability:

- A system (A, B, C) is called **observable** if any initial state $x_0 \in R^n$ can be uniquely determined given the system input trajectory $u(0), u(1), \dots, u(k-1)$ and output trajectory $y(0), y(1), \dots, y(k)$ for some finite k

- What is the relation between output and initial state?
  - Note $x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j)$

$$= A^k x_0 + Bu(k-1) + ABu(k-2) + \cdots + A^{k-1}Bu(0)$$

$$= A^k x_0 + \underbrace{[A^{k-1}B \quad A^{k-2}B \quad \cdots \quad AB \quad B]}_{m \cdot \text{columns}} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$m \times 1$ vector

$y(0) = Cx(0) + Du(0)$

$y(1) = Cx(1) + Du(1) = CAx(0) + CBu(0) + Du(1)$

$y(2) = Cx(2) + Du(2) = CA^2x(0) + CABu(0) + CBu(1) + Du(2)$

$\vdots$

$\begin{bmatrix} \vdots \end{bmatrix}$

Y_k

- Continue derivation for relation between $y(\cdot)$ and x_0

Let's define $Y_k = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix}$, $U_k = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$

unknown

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(k-1) \end{bmatrix}}_{Y_k} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}}_{O_k \in \mathbb{R}^{(k\cdot p)\times n}} x_0 + \underbrace{\begin{bmatrix} D & 0 & 0 & 0 & \cdots & 0 \\ CB & D & 0 & 0 & \cdots & 0 \\ CAB & CB & D & 0 & 0 & \cdots \\ \vdots & & & \ddots & & \\ CA^{k-2}B & CA^{k-3}B & \cdots & & CB & D \end{bmatrix}}_{T_k} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(k-1) \end{bmatrix}}_{U_k}$$

known

unknow known

- In vector form, we have: $Y_k = O_k x_0 + T_k U_k \Rightarrow O_k x_0 = (Y_k - T_k U_k)$
  - O_k maps initial state x_0 to ouput over time $[0, k-1]$
  - T_k maps input to output over time $[0, k-1]$

suppose

For example: $O_k x_0 = b$

$\forall \tilde{x} \in Null(O_k)$

- $Null(O_k)$ gives ambiguity in determining x_0

$O_k \cdot \tilde{x} = 0 \Rightarrow O_k(x_0 + \tilde{x}) = b$

- x_0 can be uniquely determined if $Null(O_k) = \{0\}$, i.e. $\boxed{rank(O_k) = n}$, for some finite k

Recall: conservation of dim

$$\dim(Null(O_k)) + rank(O_k) = n$$

n: # of columns

- Input u does not affect **ability** to determine x_0
  - its effect can be subtracted out
  - So we often say system (A, C) is observable or not (No need to mention B)

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}, \quad k \nearrow$$

- **Observability Test:** (A,C) is observable if and only if observability matrix M_o is full rank, where $M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$
  - Note: (A,C) observable $\Leftrightarrow$ Any x_0 can be uniquely determined by $u(0), \dots, u(k-1), y(0), \dots, y(k-1)$, for a finite k

n = state dim

why? According to the previous slide, x_0 can be uniquely determined if $\text{rank}(\mathcal{O}_k) = n$, for some finite k

By "Fact": (CH theorem on slide 12)

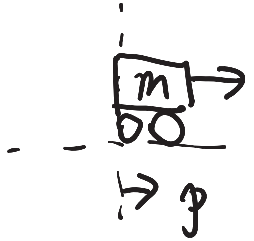
$$\text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right), \text{ for } k \geq n$$

$$\text{rank}\left(\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \\ CA^{n} \end{bmatrix}\right) \dots\dots \Rightarrow \text{sys observable} \Leftrightarrow \text{rank}(M_o) = n$$

- Observability example:

$$\cancel{A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, y(k) \equiv 2, u(k) \equiv 0, k = 0,1,2}$$

Example:



dynamics: $\underline{m\ddot{p} = u}$

Let's define $x_1 = p$

$x_2 = \dot{p}$

$\Rightarrow$ state space model:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

option ① : $y = [1 \quad 0]\, x$ $\Leftarrow$ measure position

② : $y = [0 \quad 1]\, x$ $\Leftarrow$ measure velocity

$\Downarrow$ discretize with Δt "Euler"

$$x(k+1) = (I + \Delta t \cdot A)\, x(k) + \Delta t\, B\, u(k) = \underbrace{\begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ \frac{\Delta t}{m} \end{bmatrix}}_{B} u(k)$$

① $y(k) = \underline{[1 \quad 0]}\, x(k)$ $\Leftarrow$ observable ?

② $y(k) = [0 \quad 1]\, x(k)$ $\Leftarrow$ observable ?

- For case 1: $n=2$

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \Delta t \end{bmatrix}$$

$\text{rank}(M_o) = 2$

$\Rightarrow$ observable

For case 2:

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$\text{rank}(M_o) = 1$

$\Rightarrow$ unobserable

try to have more rows

$$O_K = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}$$

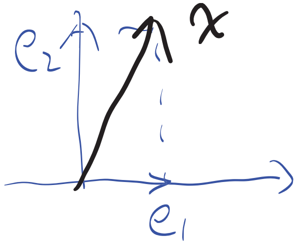
doesn't have to have more measurement

Outline

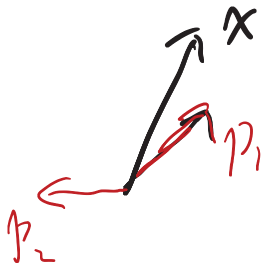
- State Space Solutions
- Internal Stability
- Controllability
- Observability
- **Invariance under Similarity Transformation**

- Change of basis vectors for equivalent system representation
  - Canonical coordinate system:

$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \cdot e_1 + 3 e_2$



  - New coordinate system: basis vectors



e.g. we can choose new basis vectors in $\mathbb{R}^2$

$p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

new coordinate of x wrt $[p_1, p_2]$

$x = 2e_1 + 3e_2 = 3p_1 + p_2 = [p_1 \; p_2] \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

  - Relation: $x = Pz$

P is nonsingular matrix

$P = [p_1 \; p_2]$

$\underbrace{[e_1 \; e_2]}_{I} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [p_1 \; p_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

- Dynamics in new coordinate systems:
  - $x(k) = Pz(k) \Rightarrow z(k) = P^{-1}x(k)$
  - Dynamics in original coordinate:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k),$$

  - Dynamics in the new coordinate

$$z(k+1) = \hat{A}z(k) + \hat{B}u(k), \; y(k) = \hat{C}z(k) + \hat{D}u(k)$$

  - Derive $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$

$$z(k+1) = P^{-1}x(k+1) = P^{-1}Ax(k) + P^{-1}Bu(k)$$

$$= \underbrace{P^{-1}AP}_{\hat{A}}z(k) + \underbrace{P^{-1}B}_{\hat{B}}u(k)$$

$$y(k) = \underbrace{CP}_{\hat{C}}z(k) + \underbrace{D}_{\hat{D}}u(k)$$

$$\boxed{\begin{aligned} \hat{A} &= P^{-1}AP \\ \hat{B} &= P^{-1}B \\ \hat{C} &= CP \\ \hat{D} &= D \end{aligned} \quad (1)}$$

  - System $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is called **equivalent** to original system (A, B, C, D)

$\hat{\phi}$ defined in (1) for any nonsingular P

- Similarity transformation does not change system stability, controllability and observability
  - A and PAP^{-1} have the same Jordan form, so similarity transformation does not change stability, i.e., **A is (asymptotically) stable iff $\hat{A}$ is (asymptotically) stable**

General case: $A = T \underbrace{J}_{\text{Jordan form}} T^{-1}$

$$\hat{A} = PAP^{-1} = PTJT^{-1}P^{-1} = (PT)J\cdot(PT)^{-1}$$

$\hat{A} \sim A$ has the same set of eigenvalues

- $\widehat{M}_c = P^{-1}M_c$, with this it can be shown that M_c full rank iff $\widehat{M}_c$ full rank

therefore, $(\boldsymbol{A}, \boldsymbol{B})$ **controllable iff** $(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{B}})$ **controllable**

$$\hat{M}_c = [\hat{B} \quad \hat{A}\hat{B} \quad \cdots \quad \hat{A}^{n-1}\hat{B}]$$

$$= [P^{-1}B \quad P^{-1}A\cancel{P}\cancel{P^{-1}}B \quad \cdots\cdots \quad (P^{-1}AP)^{n-1}P^{-1}B]$$

$$= P^{-1}[B \quad AB \quad \cdots \quad A^{n-1}B] = P^{-1}M_c$$

V.F.Y.: if P is nonsingular, $\operatorname{rank}(M_c) = \operatorname{rank}(P^{-1}M_c) \leftarrow$

$\widehat{M}_o = M_o P$, similarly, this implies **(A, C) observable iff** $(\hat{\mathbf{A}}, \hat{\mathbf{C}})$ **observable**

$$\operatorname{rank}(\underbrace{\hat{M}_o}_{M_o \cdot P}) = n \iff \operatorname{rank}(M_o) = n$$