Fall 2021 ME424 Modern Control and Estimation

Lecture Note 5
State-Feedback and Output Feedback Control

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Prof. Wei Zhang
Department of Mechanical and Energy Engineering
SUSTech Institute of Robotics
Southern University of Science and Technology

zhangw3@sustech.edu.cn
https://www.wzhanglab.site/

Outline

- **Eigenvalues ↔ System Response**
- Full State-feedback: Eigenvalue Assignment
- Luenberger Observer Design
- Output-feedback Control and Separation Principle

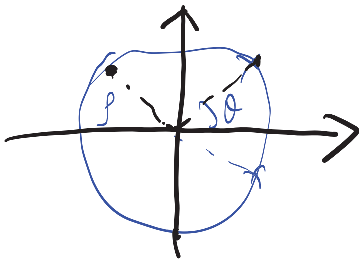
- State space solution (with zero control $u(k) = 0$)
  - $x(k) = A^k x(0)$
  - **Simple Case** (Diagonalizable):
    - $A = TDT^{-1} = T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{bmatrix} T^{-1}$
    - $D^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_q^k \end{bmatrix}$
  - Transient response depends on the terms of the form λ_i^k

- **General case: Jordan form**
  - $A = TJT^{-1} = T \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix} T^{-1} \Rightarrow \; A^k =$
  - **Fact:** if $J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \Rightarrow J_i^k = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \frac{k(k-1)}{2}\lambda_i^{k-2} \\ 0 & \lambda_i^k & k\lambda_i^{k-1} \\ 0 & 0 & \lambda_i^k \end{bmatrix}$
- Transient response depends on the terms of the form

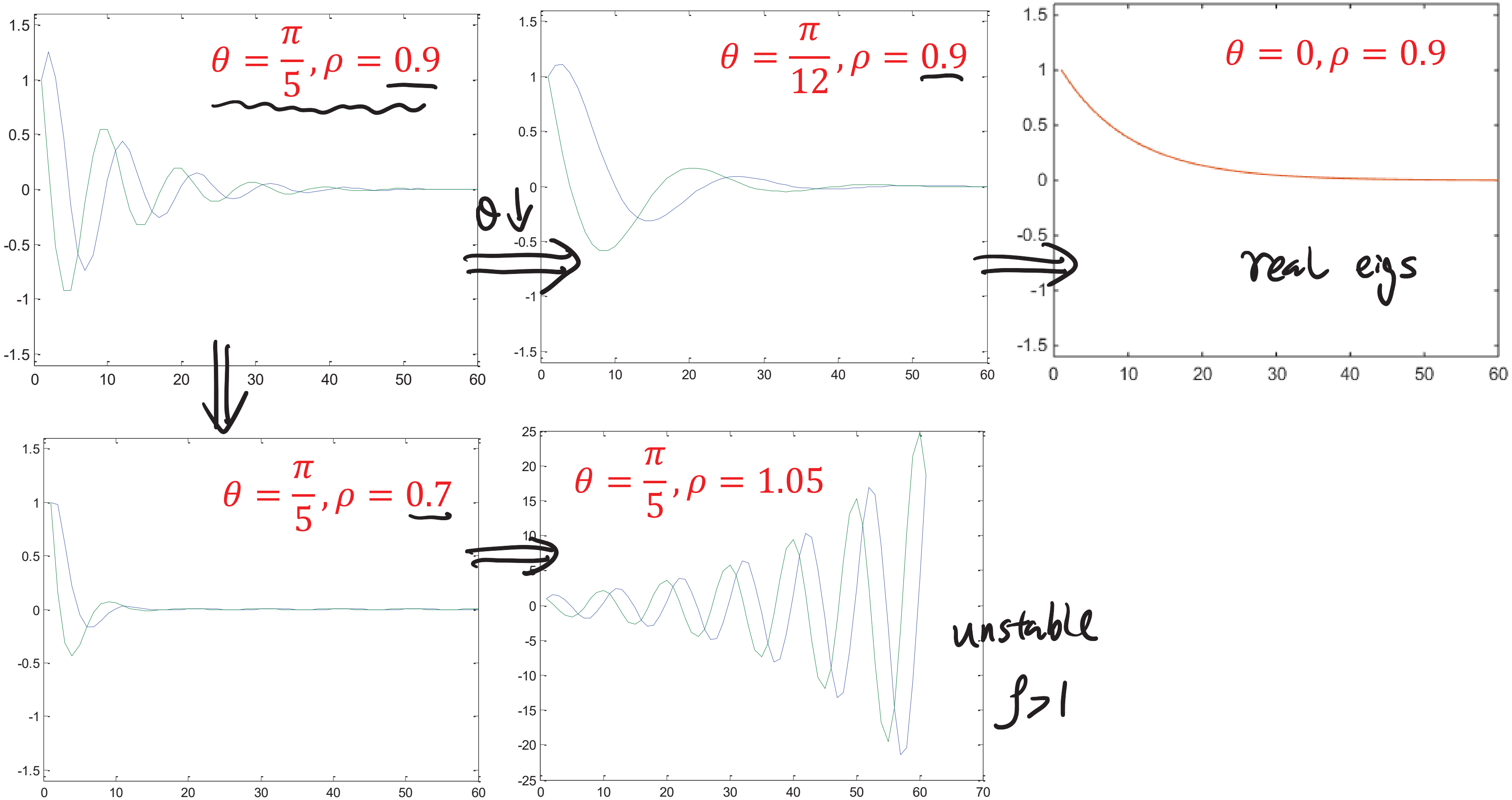
$$\frac{k(k-1)\cdots(k-j)}{j!}\lambda_i^{k-j}$$

- The shape of transient response is determined by the locations of the eigenvalues

$$A = \rho \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \Rightarrow \lambda_{1,2} = \rho(\cos(\theta) \pm j\sin(\theta))$$



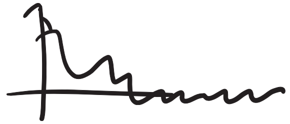
e.g: $x(k) = A^k x(0)$, with $x(0) = [1 \quad 1]^T$



- Large $|\lambda|$ produces slow convergence, while a small $|\lambda|$ produces fast convergence

$\theta = 0$

- A real λ produces a monotonic response, while a complex λ produces an oscillatory response



- For a complex λ, the response becomes more oscillatory as the ratio $\left|\frac{Im(\lambda)}{Re(\lambda)}\right|$ increases

- Control design goal (for linear system): to $\boxed{\text{modify}}$ the $\boxed{\text{eigs}}$ of original system to achieve desired response.

- Feedback control fall into two categories
  - **State Feedback**: all state variables are measured and can be used in feedback

$$u(t) = g(x(t))$$

$$x(t) \rightarrow \boxed{\text{Controller}} \rightarrow u(k)$$

  - **Output Feedback**: Only output $y = Cx + Du$ (typically dim(y)<dim(x)) are measured and can be used in feedback

$$u(t) = g(y(t))$$

$$y(t) \rightarrow \boxed{\text{controller}} \rightarrow u(k)$$

- Response under zero-input $u(k)\equiv 0,\ \forall k$.

$$A = TJT^{-1}$$

What about $u(k)\neq 0$, e.g. step response. $u(k)\equiv 1$

$$x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u(j) = \underbrace{A^k x_0}_{0} + \left(\sum_{j=0}^{k-1} \underbrace{T(J^{k-1-j})T^{-1}}_{A^{k-1-j}} B\right)\cdot 1$$

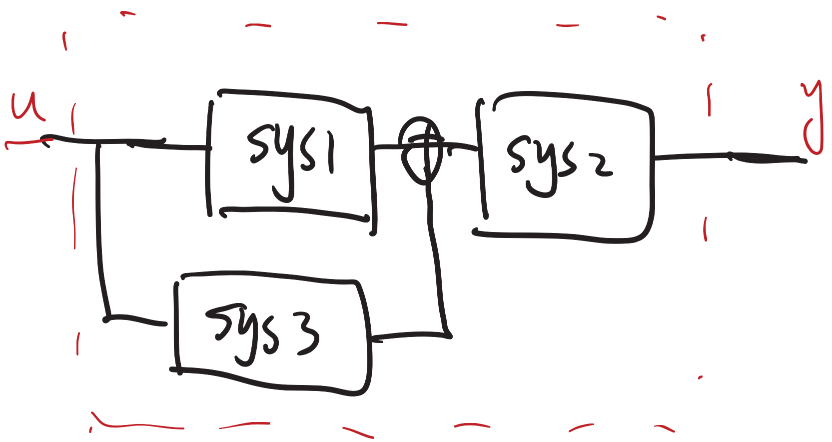
If system stable: $A^k x_0 \to 0$

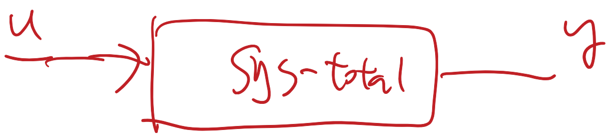
$|\lambda_i|<1$ $\quad \sum A^{k-1-j} B\cdot 1 = T\left(\sum J^{k-1-j}\right) T^{-1} B$

$$\begin{bmatrix} \sum_j \lambda_1^{k-1-j} & & \\ & \sum_j \lambda_2^{k-1-j} & \\ & & \ddots \end{bmatrix}$$

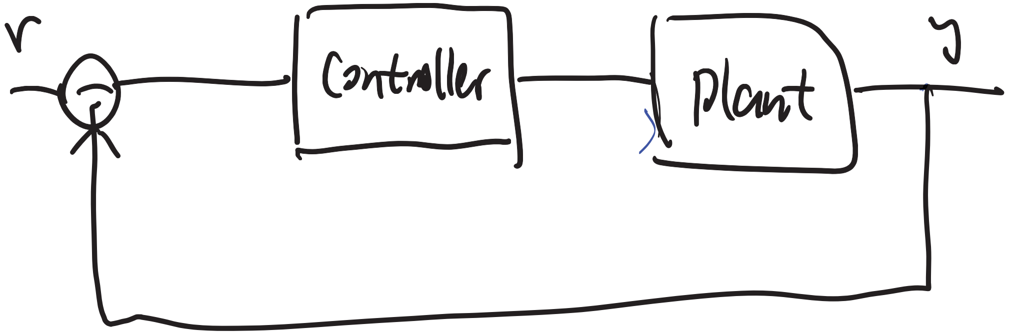
e.g. $\sum_j^k \left(\frac{1}{2}\right)^{k-1-j} \xrightarrow{k\to\infty}$ finite constant

- Closed-loop:

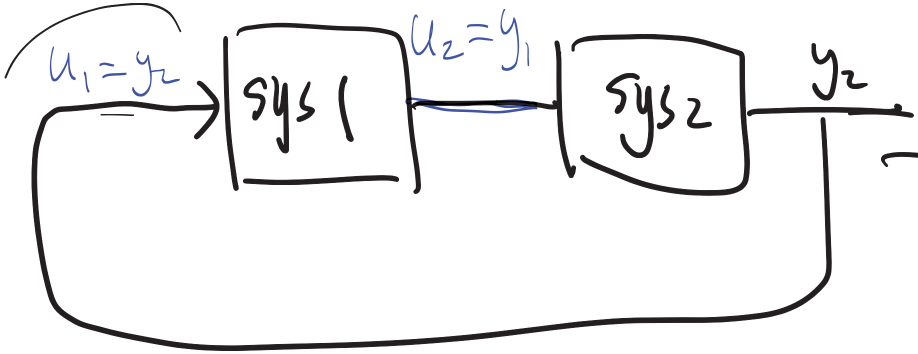




closed-loop system



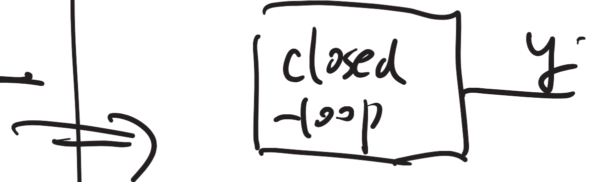
e.g.



sys1: $\begin{cases} x(k+1) = A_1 x_1(k) + B_1 u_1(k) \\ y_1(k) = C_1 x_1(k) \end{cases}$ $x_1 \in \mathbb{R}^{n_1}$

sys2: $\begin{cases} x_2(k+1) = A_2 x_2(k) + B_2 u_2(k) \\ y_2(k) = C_2 x_2(k) \end{cases}$ $x_2 \in \mathbb{R}^{n_2}$

1°: closed-loop sys model.



$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{n_1+n_2}$$

$u_1(k) = y_2(k)$

$$x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 x_1(k) + B_1 C_2 x_2(k) \\ A_2 x_2(k) + B_2 C_1 x_1(k) \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Outline

A_{CL}

$$\begin{cases} x(k+1) = A_{CL} \cdot x(k) \\ y(k) = y(k) = [0 \; C_2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

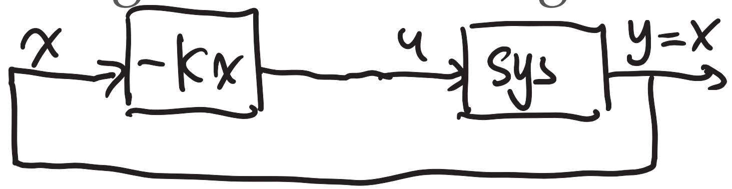
- Eigenvalues ↔ System Response
- **Full State-feedback: Eigenvalue Assignment**
- Luenberger Observer Design
- Output-feedback Control and Separation Principle

object: derive the state $x(k)$ to "zero" (stabilization/regulation)

- State feedback: full state information available to make control decision:

  e.g.: $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $u=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $K \in \mathbb{R}^{3\times 2}$

  - We focus on linear case: Let $u = -Kx$ ← linear func of state (gain)
  - we just need to design the feedback gain matrix K



- Plug in to obtain "closed-loop" system:
  - $x(k+1) = Ax(k) + Bu(k) = Ax(k) + B\cdot(-Kx(k)) = \underbrace{(A-BK)}_{\in \mathbb{R}^{n\times n}}x(k)$
  - Closed-loop system matrix: (A-BK) $= A_c$

    $x(k+1) = (A-BK)x_k$

  - Pole placement (eigenvalue assignment) problem: find K so that the closed-loop system $A - BK$ has the desired set of eigenvalues

key: $\mathrm{eig}(A-BK) \Leftarrow$ solution to $\Delta_{A_c}(\lambda) = \underbrace{\det(\lambda I - (A-BK)) = 0}_{\text{roots are eigs of } A_c}$

- **Single Input case:** $B \in \mathbb{R}^{n \times 1}$
  - Consider **controllable canonical form**

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}, \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} \text{ and } \bar{D} \text{ arbitrary}$$

  - If a system $(\bar{A}, \bar{B})$ is in controllable canonical form, then it is always controllable (verify this by checking the controllability matrix of (A, B))

$$\bar{M}_c = [\bar{B} \;\; \bar{A}\bar{B} \;\; \cdots \;\; \bar{A}^{n-1}\bar{B}]$$

e.g. $\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\bar{M}_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ 1 & c & b+c^2 \end{bmatrix} \Rightarrow \text{rank}(\bar{M}_c) = 3 \quad \text{for any } a, b, c.$$

one can show that $\bar{M}_c$ is always full rank for any $\bar{A}, \bar{B}$ in controllable canonical form

- Characteristic polynomial for $\bar{A}$

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}$$

  - $\Delta_{\bar{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

- Characteristic polynomial for closed-loop $A_{cl} = \bar{A} - \bar{B}\bar{K}$ ← design variable; $\bar{B} \in \mathbb{R}^{n\times 1}$, $\bar{K} \in \mathbb{R}^{1\times n}$
  - Assume: $\bar{K} = [k_1, k_2, \dots, k_n]$

$$\bar{A}_{cl} = \bar{A} - \bar{B}\bar{K}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [k_1 \; k_2 \; \cdots \; k_n]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -\alpha_0 - k_1 & -\alpha_1 - k_2 & \cdots & & & -\alpha_{n-1} - k_n \end{bmatrix} \leftarrow$$

- $\Delta_{A_{cl}}(\lambda) = \lambda^n + (\alpha_{n-1} + k_n)\lambda^{n-1} + (\alpha_{n-2} + k_{n-1})\lambda^{n-2} + \cdots + (\alpha_1 + k_2)\lambda + (\alpha_0 + k_1)$

choose $k_1 \cdots k_n$ ⇒ completely determine the coefficient of $\Delta_{A_{cl}}(\lambda)$

⇒ . . . . eigs of A_{cl}

- Eigenvalue assignment: given desired $\lambda_1, \ldots, \lambda_n$, how to choose $\bar{K}$?
  - **Step 1**: Find desired closed-loop characteristic polynomial:
    - $\Delta_{desired}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n + \alpha_{n-1}^* \lambda^{n-1} + \cdots + \alpha_1^* \lambda + \alpha_0^*$

e.g. $\lambda_{desired,1} = 0.1$ then $\Delta_{desired}(\lambda) = (\lambda - 0.1)(\lambda + 0.1)$

$\lambda_{desired,2} = -0.1$

$= \lambda^2 + 0 \cdot \lambda - 0.01$

$\Rightarrow \alpha_0^* = -0.01, \ \alpha_1^* = 0$

  - **Step 2**: We know: $\Delta_{A_{cl}}(\lambda) = \lambda^n + (\alpha_{n-1} + k_n)\lambda^{n-1} + (\lambda_{n-2} + k_{n-1})\lambda^{n-2} + \cdots + (\alpha_1 + k_2)\lambda + (\alpha_0 + k_1)$

    choose $k_1, \ldots, k_n$ to match coefficients

We want to choose $\bar{K} = [k_1 \cdots k_n]$ such that

$$\Delta_{A_{cl}}(\lambda) = \Delta_{desired}(\lambda)$$

Match coefficient

$$\Longrightarrow \quad k_1 = \alpha_0^* - \alpha_0, \ k_2 = \alpha_1^* - \alpha_1, \ \ldots, \ k_n = \alpha_{n-1}^* - \alpha_{n-1}$$

- Eigenvalue assignment example:

$$\bar{A} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{desired eig: } \lambda_1^* = 0.5, \quad \lambda_2^* = -0.5$$

Find $\bar{K} = [k_1 \ \ k_2]$ such that $eig(\bar{A} - \bar{B}\bar{K}) = \{0.5, -0.5\}$

Step 1: $\Delta_{desired}(\lambda) = (\lambda - 0.5)(\lambda + 0.5) = \lambda^2 + 0 \cdot \lambda - 0.25$

$\Rightarrow \alpha_0^* = -0.25, \quad \alpha_1^* = 0$

Step 2: $\alpha_0 = 1, \quad \alpha_1 = 2 \Rightarrow k_1 = \alpha_0^* - \alpha_0 = -1.25, \quad k_2 = \alpha_1^* - \alpha_1 = -2$

$\Rightarrow \bar{K} = [-1.25 \ \ -2]$

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check: $\bar{A}_{cl} = \bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1.25 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix}$

$\det(\lambda I - \bar{A}_{cl}) \overset{\text{V.F.T.}}{=} \lambda^2 - 0.25 \Rightarrow \lambda = \pm 0.5$ ✓

SISO

- What about general single input system (A, B), with $B \in R^{n\times 1}$
  - **Recall:** If original system: $x(k+1) = Ax(k) + Bu(k)$. ← not in canonical form
    Controllability matrix: $M_c = [B\ AB \cdots\ A^{n-1}B]$ → nonsingular
  - Under similarity transformation: $x(k) = P\bar{x}(k)$, we have:
    $\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k)$, with $\bar{A} = P^{-1}AP, \bar{B} = P^{-1}B$
    $\bar{M}_c \triangleq [\bar{B}\ \bar{A}\bar{B}\ \cdots \bar{A}^{n-1}\bar{B}] = P^{-1}M_c$ ← given as $[B, AB \cdots, A^{n-1}B]$

    how to find this P?

If we can find $\bar{M}_c$, $\Rightarrow P^{-1} = \bar{M}_c M_c^{-1}$, $P = M_c \bar{M}_c^{-1}$

  - $\boxed{\text{FACT:}}$ $eig(A) = eig(\bar{A})$, $hence \Rightarrow \Delta_A(\lambda) = \Delta_{\bar{A}}(\lambda)$

    $\bar{A} = P^{-1}AP$

    given as we know A,

- Main idea:
  - transform the system into a controllable canonical form $(\bar{A}, \bar{B})$
  - Design gain $\bar{K}$ to assign $eig(\bar{A} - \bar{B}\bar{K})$ to desired ones
  - Transform back to the original coordinate to get K so that eig(A-BK) = eig$(\bar{A} - \bar{B}\bar{K})$

- Eigenvalue assignment procedure for general single input system (A, B) To find P we need $\bar{M}_c \Leftarrow$ we need $\bar{A} \Leftarrow$ low row $\bar{A} \Leftarrow$ $\Delta_{\bar{A}}(\lambda)$ $\Uparrow$ $= \underbrace{\Delta_A(\lambda)}_{\text{known}}$
  - Step 1: Similarity transform: find P, such that $x(k) = P\bar{x}(k)$, and $\bar{x}(k)$ dynamic is in controllable canonical form

(1) Given A, compute: $\Delta_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

given, can be computed

(2) We know: $\Delta_{\bar{A}}(\lambda) = \Delta_A(\lambda)$, by controllable canonical form structure, we have

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}, \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

(3) Compute controllability matrix: $\bar{M}_c$ using $(\bar{A}, \bar{B})$ and M_c using (A, B)

$$\Rightarrow \quad P = M_c \bar{M}_c^{-1}$$

- Step 2: find $\bar{K}$ to assign desired eigs for $(\bar{A}, \bar{B})$

we have learned how to find $\bar{K}$ such $\text{eig}(\bar{A}-\bar{B}\bar{K}) = \text{eig}_{\text{desired}}$

- Step 3: compute $K = \bar{K}P^{-1}$

we need $\text{eig}(A-BK) = \text{eig}_{\text{desired}}$

  - Note that $A - BK$ and $\bar{A} - \bar{B}\bar{K}$ have the same set of eigs

$$\bar{A}-\bar{B}\bar{K} = P^{-1}AP - P^{-1}BKP = P^{-1}(A-BK)P$$

- Coding Example: A =[2 0 -2; 4 -2 2; 0 2 -2], B = [1 0 1]′;

- What about multiple inputs: ($B \in \mathrm{R}^{n\times m}, m \geq 2$)
  - Sometimes has redundancy, we can just use one column of B to assign eigs

we have learned how to design

e.g. $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, $B=\begin{bmatrix}1\\0\\2\end{bmatrix}$, single input $\Rightarrow$ $K=[k_1\ k_2\ k_3]$ such that

$eig(A-BK)= eig_{desired}$

Ex 1: $\tilde{A}=[\text{same}]$, $\tilde{B}=\begin{bmatrix}1&1\\0&1\\2&0\end{bmatrix}$, Can you find $\tilde{K}=\begin{bmatrix}x&x&x\\x&x&x\end{bmatrix}$, $u=-Kx$

The sln is not unique. e.g. we can pick: $\tilde{K}=\begin{bmatrix}k_1&k_2&k_3\\0&0&0\end{bmatrix} \Rightarrow eig(\tilde{A}-\tilde{B}\tilde{K})$

$=eig(A-BK)$

Ex 2: $\hat{A}=[\text{same}]$, $\hat{B}=\begin{bmatrix}1&0\\0&0\\1&1\end{bmatrix}$, we can let $\hat{K}=\begin{bmatrix}k_1&k_2&k_3\\k_1&k_2&k_3\end{bmatrix}$

  - General case is quite involved, use numerical tools to assign eigs or use LQR controller which will be covered later

$eig(\hat{A}-\hat{B}\hat{K})= eig_{desired}$

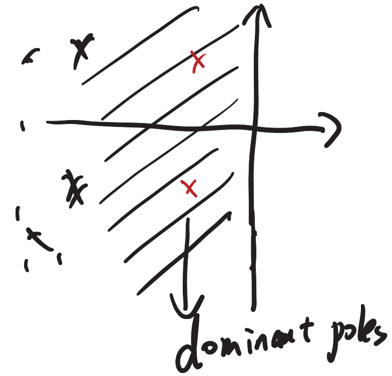
$\hat{A}-\begin{bmatrix}1&0\\0&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}k_1&k_2&k_3\\k_1&k_2&k_3\end{bmatrix}$

$= A-\begin{bmatrix}k_1&k_2&k_3\\0&0&0\\2k_1&2k_2&2k_3\end{bmatrix}$

$A-BK$

$=A-\begin{bmatrix}1\\0\\2\end{bmatrix}[k_1\ k_2\ k_3] = A-\begin{bmatrix}\ \end{bmatrix}$

- Remarks on choosing desired poles (eigenvalues)
  - Continuous time case:



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - ω_n : natural frequency
  - ζ: damping ratio ($\zeta > 1$ overly-damped, $\zeta = 1$ critically damped, $\zeta < 1$ under-damped)

$$H(s) = \frac{N(s)}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

$p_i = \sigma_i \pm j\omega_i$

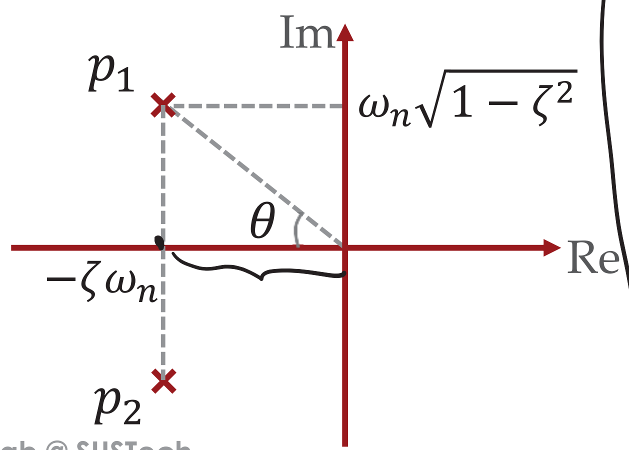
  - Under damped system ($\zeta < 1$): two complex poles:

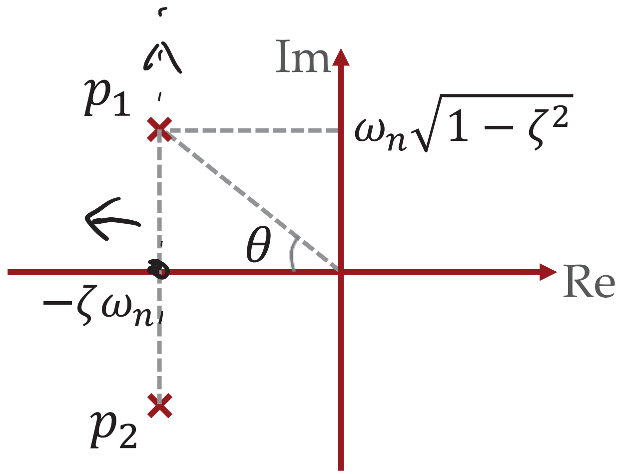
$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, define: $\theta = \cos^{-1}\zeta$

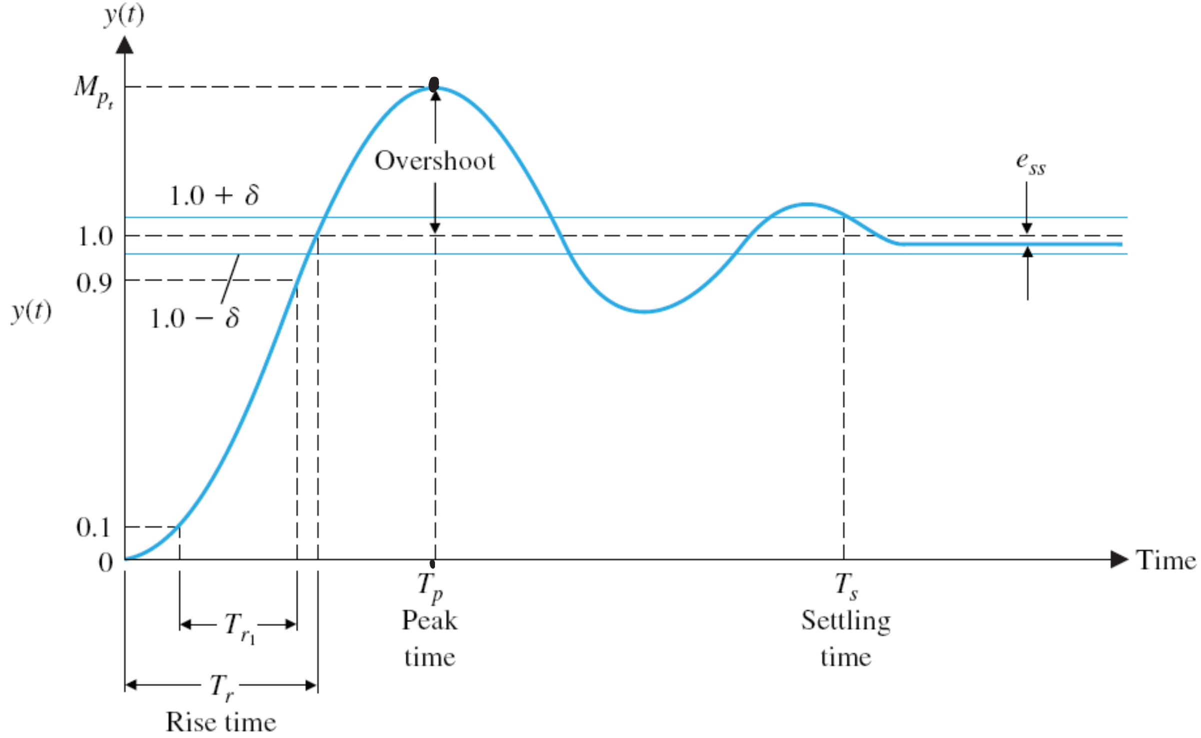
$\sigma_3 \ll \sigma_2 = \sigma_1$

$e^{(\sigma_3 \pm j\omega_3)\cdot t}$

$$\cos\theta = \frac{\zeta\omega_n}{\sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}} = \zeta$$







$T_s \approx \frac{4}{\zeta\omega_n}$, **settling time**

$T_p \approx \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$, **peak time**

PO$= 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$ **percent overshoot**

- Tradeoffs:

1. Poles moves to the left, i.e. larger $\zeta\omega_n$

$\Rightarrow$ $\zeta\omega_n \uparrow$ bigger $\Rightarrow$ T_s decreases

2. Poles moves up, i.e., larger $\omega_n\sqrt{1-\zeta^2}$

$\Rightarrow$ smaller T_p (but large $\theta \Rightarrow$ larger P.O.)

3. Smaller θ $\Rightarrow$ smaller P.O.

key contraint (implicit): we can't move poles too much to the left (typically that results in large control action)

- Discrete time case:
  - Relations:
    - $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
    - T: sampling time
    - $z = e^{sT}$

z-transform $\Leftrightarrow$ Laplace transform

$\Downarrow$

poles: z_1, z_2 $\leftarrow$ poles: p_1, p_2

$z_1 = e^{p_1 T}$, $z_2 = e^{p_2 T}$

- Pole selection example:
  - Suppose we want settling time $T_s \leq 5$ sec and $PO \leq 35\%$

requirements $\Rightarrow \begin{cases} \dfrac{4}{\zeta\omega_n} \leq 5 \\ 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq 35 \end{cases} \Rightarrow \begin{cases} \zeta\omega_n \geq 0.8 \\ \zeta \geq 0.32 \end{cases}$

We can choose $\zeta = 0.5$ $\zeta\omega_n = 1$ $\Rightarrow$ $\omega_n = 2$

$\Rightarrow p_{1,2} = -1 \pm j\sqrt{3}$ If sampling time $T = 0.03$

$z_{1,2} = e^{(-1 \pm \sqrt{3}j)\cdot 0.03}$

Outline

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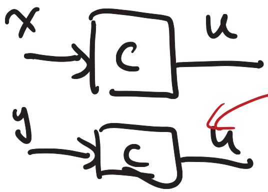
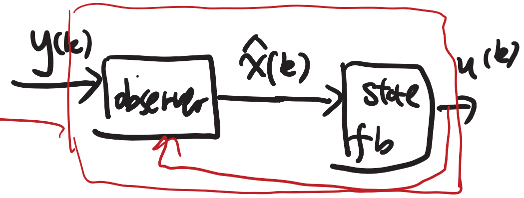
- Full State-feedback: Eigenvalue Assignment
- **Luenberger Observer Design**
- Output-feedback Control and Separation Principle

Full state feedback





- **Observer Design:** Output feedback
  - state vector is not available; u can only depend on output y
  - **Observer:** estimate system state vector $\hat{x}(k) \approx x(k)$ given $y(k)$, $u(k)$ and (A, B, C, D)
  - **Key**: generate estimate iteratively according to known system dynamics:

estimate at k — correction term

$$\hat{x}(k+1) = \underbrace{A\hat{x}(k) + Bu(k)}_{} + \boldsymbol{L}\left[\boldsymbol{y(k)} - \overbrace{\boldsymbol{C\hat{x}(k)} - \boldsymbol{Du(k)}}^{\text{correction term}}\right]$$

If $\hat{x}(k) = x(k)$, then the correction term $= 0$, $\Rightarrow \hat{x}(k+1) = x(k+1)$

$\hat{x}(k) \neq x(k)$, then modify the estimate by multiplying gain matrix L with error $y(k) - \hat{y}(k)$ $\qquad \hat{y}(k) = C\hat{x}(k) + Du(k)$

  - Iteratively update state estimate using previous estimate $\hat{x}(k)$ and new data available at time k: $u(k), y(k)$,
  - This way of estimating state is called Luenberger observer

$A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $L \in \mathbb{R}^{n\times p}$

- State estimation error vector: $e(k) = x(k) - \hat{x}(k)$ $\in \mathbb{R}^n$
- Error dynamics: $e(k+1) = (A - LC)e(k)$

$$e(k+1) = x(k+1) - \hat{x}(k+1) = Ax(k) + \cancel{Bu(k)} - \left(A\hat{x}(k) + \cancel{Bu(k)} + L\left(y(k) - C\hat{x}(k) - \cancel{Du(k)}\right)\right)$$

Note : $y(k) = Cx(k) + D\cancel{u(k)}$

$$= A(\underbrace{x(k) - \hat{x}(k)}_{e(k)}) - LC(\underbrace{x(k) - \hat{x}(k)}_{e(k)})$$

$$\Rightarrow \quad e(k+1) = (A - LC)\cdot e(k)$$

known

- **Goal**: design L matrix such that $eigs(A - LC)$ are at desired locations to ensure estimation error converge to zero with a desired transient

- **Observer design**: Find observer gain matrix ***L*** such that error dynamics have desired eigenvalues

$$x_{k+1} = A^T x_k + C^T \cdot u_k$$

- **Duality Theorem**: (A, C) observable $\Leftrightarrow$ (A^T, C^T) controllable

(remark: We say a pair (F, H) is controllable if a system with "A" matrix equal to F and "B" matrix equal to H is controllable. This also means $M_c = [H \quad FH \quad F^2H \quad \cdots \quad F^{n-1}H]$ is full rank)

$$x_{k+1} = F x_k + H \cdot u_k$$

(A, C) observable $\Leftrightarrow$ $\text{rank}(M_o) = n$, where $M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$\Leftrightarrow$ $\text{rank}(M_o^T) = n$

$$M_o^T = [\underbrace{C^T}_{B} \;\; \underbrace{A^T}_{A}\underbrace{C^T}_{B} \;\; \cdots \;\; (A^T)^{n-1} C^T]$$

$\Leftrightarrow$ $\boxed{(A^T, C^T) \text{ controllable}}$

- Consequence of the duality theorem: **If system (A,C) is observable**, we can use feedback gain design method to find observer gain L such that $eig(A - LC)$ has desired eigs

Goal: Find L such that $eig(A-LC) = eig_{desired}$

Let $\tilde{A} = A^T$, $\tilde{B} = C^T$, we know how to find $\tilde{K}$ such that $eig(\tilde{A}-\tilde{B}\tilde{K}) = eig_{desired}$

we also know $eig(\tilde{A}-\tilde{B}\tilde{K}) = eig((\tilde{A}-\tilde{B}\tilde{K})^T)$

$= eig(\tilde{A}^T - \tilde{K}^T\tilde{B}^T)$

$= eig(A - \tilde{K}^T C)$

So we can choose $L = \tilde{K}^T$

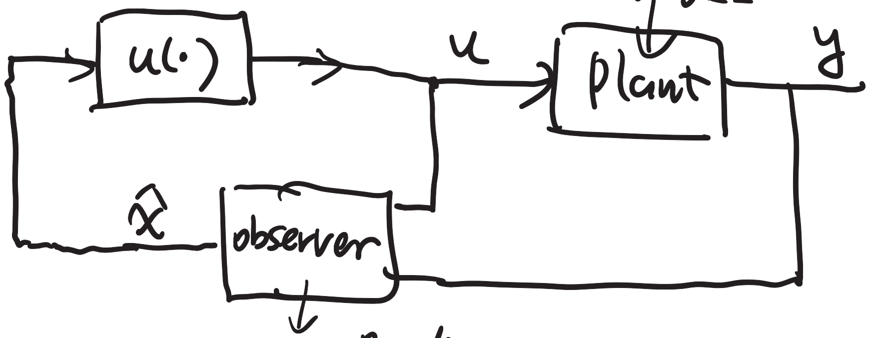
Outline

- Eigenvalues $\leftrightarrow$ System Response
- Full State-feedback: Eigenvalue Assignment
- Luenberger Observer Design
- **Output-feedback Control and Separation Principle**

- Output feedback control procedure:
  - System: $x(k+1) = A\,x(k) + Bu(k), y(k) = Cx(k) + Du(k)$
  - Find K, L such that $\underline{A - BK}$ and $A - LC$ have desired eigs
  - At time $k = 0$, pick arbitrary $\hat{x}(0)$
  - For $k \geq 0$, given $\hat{x}(k), u(k), y(k)$, compute:
    - $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L\,[y(k) - C\hat{x}(k) - Du(k)]$ $\in \mathbb{R}^n$
    - $u(k+1) = -K\hat{x}(k+1)$

$x_{k+1} = Ax_k + Bu_k$

$y_k = \cdots$



$\mathbb{R}^n$- dim dyn sys

- General guideline:

  eigenvalues of (A - BK) are chosen to meet the design specifications on the transient response. The eigenvalues of (A-LC) are chosen **much faster** than those of (A - BK)

- **Separation principle**: Observer eigs and controller eigs can be assigned separately

  $\begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$ equiv

  - Closed-loop dynamics: joint state vector: $\begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$ with $u(k) = -K\hat{x}(k)$
  - Dynamics for joint state vector:

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} Ax(k) + Bu(k) \\ Ax(k) + Bu(k) - (A\hat{x}(k) + Bu(k) + L\,[y(k) - C\hat{x}(k) - Du(k)]) \end{bmatrix}$$

$$= \begin{bmatrix} Ax(k) + B(-K)\hat{x}(k) \\ (A - LC)e(k) \end{bmatrix} = \begin{bmatrix} Ax(k) - BKx(k) + BKe(k) \\ (A - LC)e(k) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A - BK & BK \\ \hline 0 & A - LC \end{array}\right] \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$$

$$\text{eig}\left(\left[\begin{array}{c|c} \boxed{1} & * \\ \hline 0 & \boxed{2} \end{array}\right]\right) = \text{eig}([1]) \cup \text{eig}([2])$$

  - The design of K and L can be done separately to meet specified controller and observer performance (characterized by eigs(A-BK) and eigs(A-LC))

- Summary