Fall 2021 ME424 Modern Control and Estimation

Lecture Note 7: Kalman Filter - Probability Review

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Kalman Filer Preview:

- Given stochastic linear system described by

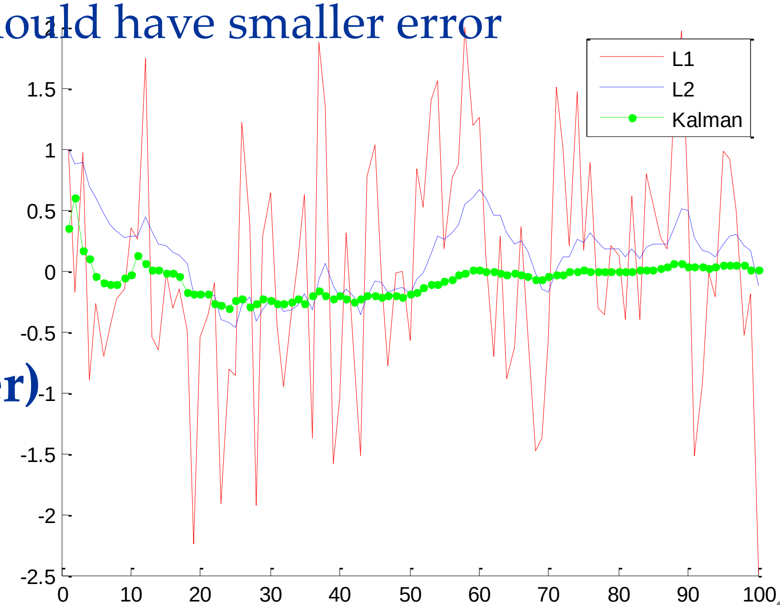
$$x_{k+1} = A_k x_k + B_k u_k + w_k$$
$$y_k = C_k x_k + D_k u_k + v_k$$

- **Kalman filter**: compute the "best" estimate of x_k given input-output data history $\{u_j, y_j\}_{j=0}^{k}$

- **From Luenberger to Kalman:**
  - **Deterministic to probabilistic model**
  - **Stable observer to optimal observer/filter**

Kalman Filer Preview: Luenberger observer vs. Kalman filter

- Example: $x_{k+1} = x_k, y_k = x_k + v_k,$, where v_k is white noise with $cov(v_k, v_m) = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{otherise} \end{cases}$
  - Ignoring noise, we have deterministic model $x_{k+1} = x_k,\ y_k = x_k,$
  - Luenberger type observer: $\hat{x}_{k+1} = \hat{x}_k + L(y_k - \hat{x}_k)$
  - Estimator error dynamics: $e(k+1) = (A - LC)e(k) = (1 - L)e(k)$
  - E.g.: $L_1 = 0.9$ and $L_2 = 0.1$, both provide stable error dynamics
  - According to deterministic model, L_1 should have smaller error
  - However, with noise, both L_1 and L_2 perform poorly, L_1 is worse than L_2
  - **The optimal observer (Kalman filter) is much better**



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- **Kalman Filter Solution**: $\hat{x}_k = \boldsymbol{E}(\boldsymbol{x_k} | \boldsymbol{y_0}, \boldsymbol{y_1}, \dots, \boldsymbol{y_k})$

- **Our goal: in-depth understanding of the assumptions, derivations of Kalman filter**

Outline

- **Probability and Conditional Probability**
- Random Variables and Random Vectors
- Jointly Distributed Random Vectors and Conditional Expectation
- Covariance Matrix

What is probability?

- A formal way to quantify the uncertainty of our knowledge about the physical world

- Formalism: Probability Space $(\Omega, \mathcal{F}, P)$
  - Ω : **sampling space**: a set of all possible outcomes (maybe infinite)
  - $\mathscr{F}$: **event space**: collection of events of interest (event is a subset of Ω)
  - P: $\mathcal{F} \rightarrow [0,1]$ probability measure: assign event in $\mathscr{F}$ to a real number between 0 and 1

Axioms of probability:

- $P(A) \geq 0$
- $P(\Omega) = 1$
- $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

- **Important consequences:**
  - $P(\emptyset) = 0$
  - Law of total probability: $P(B) = \sum_i^n P(B \cap A_i)$, for any partitions $\{A_i\}$ of Ω
    - Recall a collection of sets $A_1, \ldots, A_n$ is called a partition of Ω if
      - $A_i \cap A_j = \emptyset$, for all $i \neq j$ (mutually exclusive)
      - $A_1 \cup A_2 \cdots \cup A_n = \Omega$

Conditional probability

- Probability of event A happens given that event B has already occurred
  - $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- We assume $P(B) > 0$ in the above definition

- **What does it mean?**
  - Conditional probability is a probability: $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$
  - **"Conditional" means, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ the is derived from an original probability space $(\Omega, \mathcal{F}, P)$ given some event has occurred**
  - After B occurred we are uncertain only about the outcomes inside B

- Bayes rule: relate $P(A \mid B)$ to $P(B \mid A)$

$$P(A \mid B) = \frac{P(B|A)P(A)}{P(B)}$$

- Events A and B are called (statistically) independent if
  - $P(A|B) = P(A)$
  - Or equivalently: $P(A \cap B) = P(A)P(B)$

- **Example of conditional probability**: A bowl contains 10 chips of equal size: 5 red, 3 white, and 2 blue. We draw a chip at random and define the event:

A = the draw of a red or a blue chip

Suppose you are told the chip drawn is not blue, what is the new probability of A

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- **What is random variable and random vector?**
  - Deterministic variable:
  - Random variable:

How to specify probability measure

- Discrete random variable: probability mass function (pmf)

  e.g. toss a coin or die

- Continuous random variable: probability density function (pdf)

  e.g. temperature density

How to specify probability measure

- Random vector: scalar random variables listed according to certain order
- n-dimensional random vector: $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$
- Notation: We typically use capital to denote random variables (vectors) and lower case letter to denote specific values the random variable takes
- density function: $f(x),\ x \in \mathrm{R}^n$
- probability evaluation: $P(X \in A) = \int_A f(x)dx$

Expectation of a random vector $X \in R^n$:

Continuous random vector: $\boldsymbol{E(X) = \int_{R^n} x f(x) dx}$

Discrete random vector: $\boldsymbol{E(X) = \sum_x x \cdot Prob(X = x)}$

- Expectation: $E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix}$

- Examples: Let $X \in R^2$ be discrete random variable with $Prob\left(X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2}, Prob\left(X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \frac{1}{3}, Prob\left(X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{6}.$ Compute $E(X)$

Linearity of Expectation:

- Expectation of AX with deterministic constant $A \in R^{m \times n}$ matrix:

$$\boldsymbol{E}(\boldsymbol{AX}) = \boldsymbol{AE}(\boldsymbol{X})$$

- More generally, $E(AX + BY) = AE(X) + BE(Y)$

- Example: Suppose $\mathrm{X} \in R^2, Y \in R^3$, with $\mathrm{E}(X) = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$, , $\mathrm{E}(Y) = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$,

  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Compute $E(AX + BY)$

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Jointly distributed random vectors: $X \in R^n, Y \in R^m$

- Completely determined by joint density (mass) function: $(X, Y) \sim f_{XY}(x, y)$

  Compute probability:

- marginal density: $X \sim f_X(x), Y \sim f_Y(y)$, where

$$f_X(x) = \int_{R^m} f_{XY}(x,y)dy, \qquad f_Y(y) = \int_{R^n} f_{XY}(x,y)dx,$$

- Example: $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, Prob\left(X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2}, Prob\left(X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \frac{1}{3}, Prob\left(X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{6}$
  - This is joint distribution for X_1, X_2

- The conditional density: $(X, Y) \sim f_{XY}(x, y)$
  - Quantify how the observation of a value of Y, $Y = y$, affects your belief about the density of X
  - The conditional probability definition implies (nontrivially)

$$P(A \mid B) = P(A \cap B)/P(B) \Rightarrow p_{X|Y}(X = i | Y = j) = \frac{p_{XY}(X=i, Y=j)}{\sum_i p_{xy}(X=i, Y=j)}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

  - Law of total probability: $P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

$$f_X(x) = \int_{R^m} f_{X|Y}(x|y) f_Y(y) dy$$

$$f_Y(y) = \int_{R^n} f_{Y|X}(y|x) f_X(x) dx$$

  - ***X* is independent of *Y***, denoted by $X \perp Y$,

if and only if $\quad f_{XY}(x, y) = f_X(x) f_Y(y)$

- **Conditional expectation:**
  - The conditional mean of $X|Y = y$ is

$$E(X|Y = y) = \int_{R^n} x f_{X|Y}(x|y)dx$$

$$E(X|Y = y) = \sum_i i \cdot Prob(X = i|Y = y)$$

- Example 1:
  - E(X|Y = 1)

		X				
		2	3	4	5	6
Y	1	1/4	1/8	1/8		
	2		1/6	1/12	1/12	
	3			1/12	1/24	1/24

  - E(X|Y = 2)

- E(X|Y=3)

- **Example 2**: Suppose that (X, Y) is uniformly distributed on the square $S = \{(x, y) : -6 < x < 6, -6 < y < 6\}$. Find $E(Y|X = x)$.

- Law of total probability implies:
  - $E(X) = \sum_y E(X|Y = y) \cdot p_Y(Y = y)$
  - $E\big(g(X,Y)\big) = \sum_y E(g(X,Y)|Y = y) \cdot p_Y(Y = y)$

- Continue Example 1:

		X				
		2	3	4	5	6
Y	1	1/4	1/8	1/8		
	2		1/6	1/12	1/12	
	3			1/12	1/24	1/24

- Example 3.: outcomes with equal chance: (1,1), (2, 0), (2,1), (1,0), (1,-1), (0,0), with $g(X, Y) = X^2Y^2$

  Method 1: $E\big(g(X,Y)\big) = E(X^2Y^2) = 1^2 \cdot (-1)^2 \cdot \frac{1}{6} + 1^2 \cdot 1^2 \cdot \frac{1}{6} + 2^2 \cdot 1^2 \cdot \frac{1}{6} = 1$

  Method 2: conditioning on values of $Y = -1, 0, 1$

		X		
		0	1	2
	-1	0	1/6	0
Y	0	1/6	1/6	1/6
	1	0	1/6	1/6

- Covariance (Random variable case):
  - $Cov(X,Y) = E\left(\left(X - E(X)\right)(Y - E(Y))\right)$

		X = 1	X = 2
Y	-1	0.25	0.1
	1	0	0.65

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- **Covariance (Random variable case):**
  - If $Cov(X,Y) > 0$, X and Y are positively correlated
    - If you see a realization of X larger than $E(X)$, it is more likely for Y to be also larger than $E(Y)$
  - If $Cov(X,Y) < 0$, X and Y are negatively correlated
    - If you see a realization of X larger than $E(X)$, it is more likely for Y to be smaller than $E(Y)$
  - If $Cov(X,Y) = 0$, X and Y are uncorrelated

- **Covariance Matrix: $\boldsymbol{X} \in \boldsymbol{R}^{\boldsymbol{n}}$, $\boldsymbol{Y} \in \boldsymbol{R}^{\boldsymbol{m}}$**

$$Cov(X,Y) = E\left(\left(X - E(X)\right)\left(Y - E(Y)\right)^T\right)$$

  - It is a $n \times m$ matrix: with $(Cov(X,Y))_{ij} = \mathrm{Cov}\left(\mathrm{X_i}, \mathrm{Y_j}\right) = \mathrm{E}\left((\mathrm{X_i} - \mathrm{E}(\mathrm{X_i}))\left(\mathrm{Y_j} - \mathrm{E}\left(\mathrm{Y_j}\right)\right)\right)$

$$cov(X,Y) = \begin{bmatrix} cov(X_1,Y_1) & cov(X_1,Y_2) & \dots & cov(X_1,Y_m) \\ cov(X_2,Y_1) & cov(X_2,Y_2) & \dots & cov(X_2,Y_m) \\ \vdots & \vdots & \ddots & \vdots \\ cov(X_n,Y_1) & cov(X_n,Y_2) & \dots & cov(X_n,Y_m) \end{bmatrix}$$

- **Properties of Covariance**
  1. $Cov(X+a, Y+b) = Cov(X,Y)$
  2. $Cov(X,Y) = Cov(Y,X)^T$
  3. $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$
  4. $Cov(AX, BY) = ACov(X,Y)B^T$
  5. If $X \perp Y$, $Cov(X,Y) = 0$
  6. $Cov(X) \triangleq Cov(X,X)$ is positive semidefinite (p.s.d.)

- **Example**: Suppose you know $cov(X,Y) = \Sigma_{XY}$, $cov(X) = \Sigma_X$, $cov(Y) = \Sigma_Y$, what is $Cov(AX + BY)$?

- **Example:** Given that $E(Z) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $Cov(Z, Z) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 8 \end{bmatrix}$. Let $P = \begin{bmatrix} Z_2 \\ Z_1 \end{bmatrix}$, $Q = Z_3$

  Compute: $Cov(P, Q)$, $Cov(Q, 2P)$

- More discussions