

Fall 2021 ME424 Modern Control and Estimation

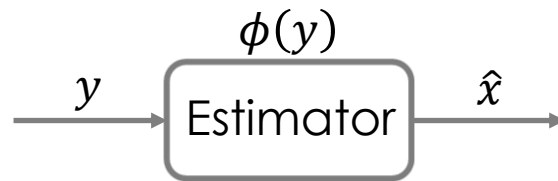
Lecture Note 7: Kalman Filter
- Extended Kalman Filter

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Recall:

- Suppose we want to estimate the value of a hidden random vector $X \in \mathbb{R}^n$ based on observations of a related vector $Y \in \mathbb{R}^m$.
- We have to know the relationship between X and Y . Suppose we take probabilistic viewpoint of their relations, namely, $(X, Y) \sim f_{XY}(x, y)$
- An estimator $\phi(y)$ is a function that maps each measurement $Y = y$ to an estimate \hat{x}



- **MMSE Theorem:** The Minimum Mean-Squared Estimator for X given $Y = y$, that minimizes $E \left(\|\phi(Y) - X\|^2 \right)$ is given by
$$\hat{X}_{MMSE} = \phi_{MMSE}(y) = E(X|Y = y)$$

Recall:

- Kalman filter is a recursive way to compute $E(x_k|Y_k)$ for linear Gaussian system
- For nonlinear systems, we can use Extended Kalman Filter (EKF)
 - System setup:
$$\begin{aligned}x_{k+1} &= f(x_k, u_k) + w_k \\ y_k &= h(x_k, u_k) + v_k\end{aligned}$$
 - $x_k \in R^n$ --- system state at time k
 - $y_k \in R^m$ --- measurement vector at time k
 - $Y_k \triangleq [y_0^T \ y_1^T \ \dots \ y_k^T]^T$ --- collection of measurements up to time k
 - $u_k \in R^p$ --- system input at time k (deterministic input)
 - $w_k \in R^n \sim N(0, Q_k)$, $v_k \in R^p \sim N(0, R_k)$
 - Assume $x_0 \sim N(\mu_0, \Phi_0)$, $x_0 \perp w_k$, $x_0 \perp v_k$, $w_k \perp v_j, \forall k, j$, and
$$w_k \perp w_j, \quad v_k \perp v_j, \quad \forall k \neq j$$

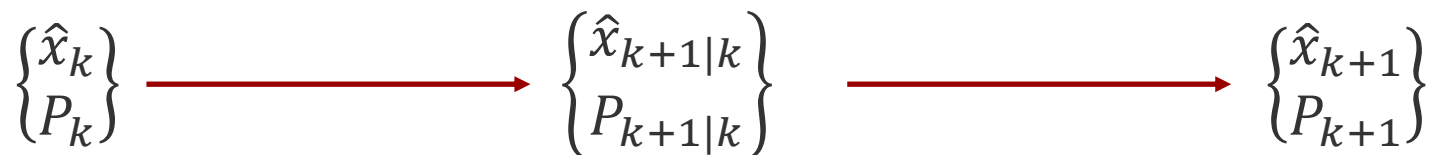
Preview of Extended Kalman Filter

- By fundamental theorem of estimation, we know that the MMSE is given by $E(x_k|Y_k)$
 - So we again needs to compute $E(x_k|Y_k)$
- With nonlinear dynamics, x_k is a random variable that **may not be Gaussian.**
- Extended Kalman Filter tries to
 - Approximate x_k as a Gaussian
- Approximate the nonlinear dynamics as linear dynamics

- Notations: $\hat{x}_{k|k} = E(x_k|Y_k)$, $\hat{x}_{k|k-1} = E(x_k|Y_{k-1})$
 $P_{k|k} = E\left((x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T \middle| Y_k\right)$
 $P_{k|k-1} = E\left((x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T \middle| Y_{k-1}\right)$

Simplified notation: $\hat{x}_k \triangleq \hat{x}_{k|k}$, $P_k = P_{k|k}$

- Goal: recursively compute:



Extended Kalman Filter Derivation:

■ Step 1: Prediction (via linearization):

- Given $\hat{x}_k = E(x_k|Y_k)$, $P_k = E((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | Y_k)$,
- Need: $\hat{x}_{k+1|k} = E(x_{k+1}|Y_k)$, $P_{k+1|k} = E\left((x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T | Y_k\right)$
- Recall the linear Gaussian case: $x_{k+1} = Ax_k + Bu_k + w_k$,
the prediction step: $\hat{x}_{k+1|k} = A_k\hat{x}_k + B_k u_k$, $P_{k+1|k} = A_k P_k A_k^T + Q_k$
- EKF: Linearize $f(x, u)$ around the current state estimate \hat{x}_k and input u_k

Summary of EKF Prediction Step:

Linearization: $F_k \triangleq \left. \frac{\partial f}{\partial x} \right|_{\hat{x}_k, u_k}$

$$\hat{x}_{k+1|k} = f(\hat{x}_k, u_k), \quad P_{k+1|k} = F_k P_k F_k^T + Q_k$$

Extended Kalman Filter Derivation:

- Step 2: Measurement update through linearization:

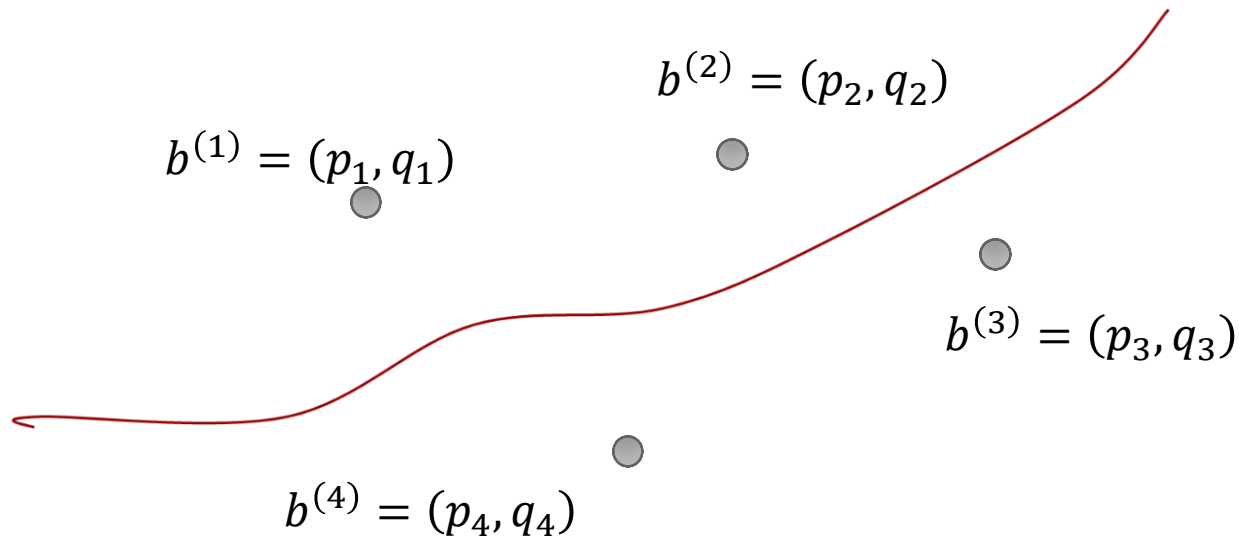
Recall the linear case:

$$\begin{aligned}x_{k+1} &= Ax_k + Bu_k + w_k \\y_k &= Cx_k + Du_k + v_k\end{aligned}$$



$$\begin{aligned}K_{k+1} &= P_{k+1|k}C^T(CP_{k+1|k}C^T + R_{k+1})^{-1} \\ \hat{x}_{k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k} - Du_{k+1}) \\ P_{k+1} &= (I - K_{k+1}C)P_{k+1|k}\end{aligned}$$

Application Example I for EKF



- beacons with known positions $b^{(i)} = (b_1^{(i)}, b_2^{(i)})$
- p_k : robot location at time k
- $y_{k,i}$: range measurement from beacon i at time k .
 - Typical measurement model: $y_{k,i} = \|b^{(i)} - p_k\| + v_i$
- Goal: find the best estimate of p_k given measurement $\{y_0, y_1, \dots, y_k\}$

- Derivation of the system model under constant speed assumption

Here, we want to use dynamics information in addition to the beacon measurement. we assume constant speed motion model:

- **EKF derivation and implementation**

Application Example II : Joint State and Parameter Estimation

- Consider a 2nd-order continuous time system:

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2y(t) = \omega_n^2 u(t)$$

- System state: $x = [y, \dot{y}]^T$, system parameter $\theta = [\xi, \omega_n]^2$
- System input-output u, y
- Question: How to use (u, y) data to jointly estimate x and θ ?

