MEE5114 Advanced Control for Robotics

Lecture 10: Basics of Stability Analysis

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Outline

This lecture introduces basic concepts and results on Lyapunov stability of nonlinear systems.

- Background
- Lyapunov Stability Definitions
- Lyapunov Stability Theorem
- Lyapunov Stability of Linear Systems
- Converse Lyapunov Function
- Extension to Discrete-Time System

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What is Stability Analysis?

- system asymptotic behavior (not too much about transient)
- ability to return to the desired asymptotic behavior (not just convergence)

General ODE Models for Dynamical Systems

- ODE: $\dot{x} = f(x, u)$, with $x(0) = x_0$
  - $x \in \mathcal{X} \subseteq \mathbb{R}^n$: state
  - $u \in \mathcal{U} \subseteq \mathbb{R}^m$: control input
  - $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$: (time-invariant) vector field

- System output $y = g(x, u)$

- Control law: $\mu : \mathcal{X} \to \mathcal{U}$

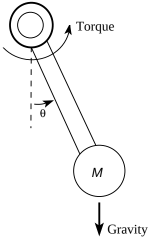
- Closed-loop dynamics under μ: $\quad \dot{x} = f(x, \mu(x))$

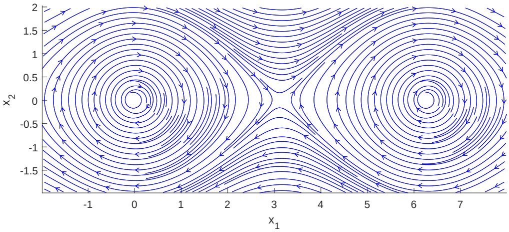
- Autonomous system:

$$\dot{x} = f(x), \text{ with } x(0) = x_0 \tag{1}$$

Example: Pendulum

- Pendulum with driving force: $\ddot{\theta} = \frac{-\rho}{Ml^2}\dot{\theta} + \frac{\cos\theta}{Ml}u + \frac{g}{l}\sin\theta$

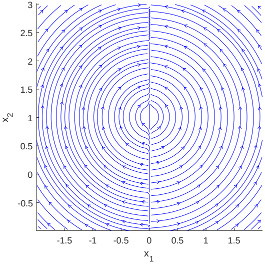




Examples: Adaptive Control

- Closed-loop dynamics under adaptive control:

$$\begin{cases} \dot{y} = y + u \\ u = -ky, \dot{k} = y^2 \end{cases}$$

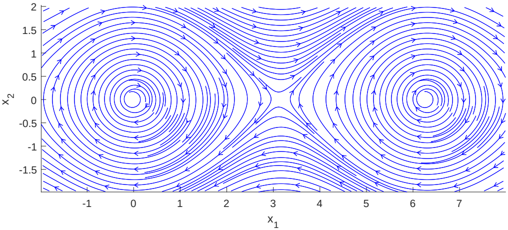


Equilibrium Point of Dynamical Systems

Definition 1 (Equilibrium Point).

A state x^* is an *equilibrium point* of system (1) if once $x(t) = x^*$, it remains equal to x^* at all future time.

- Mathematically: $f(x^*) = 0$
- E.g undamped pendulum with no driving force:



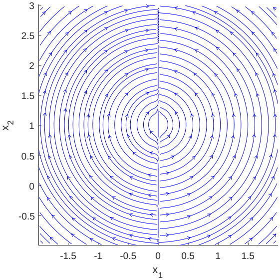
Invariant Set of Dynamical Systems

Definition 2 (Invariant Set).

A set E is an *invariant set* of system (1) if every trajectory which starts from a point in E remains in E at all future time.

- Mathematically: If $x(t_0) \in E$, then $x(t) \in E$, $\forall t \geq t_0$
- E.g: closed-loop dynamics under adaptive control:

$$\begin{cases} \dot{y} = y + u \\ u = -ky, \dot{k} = y^2 \end{cases}$$



Outline

Lyapunov Stability Definitions (1/2)

Consider a time-invariant autonomous (with no control) nonlinear system:

$$\dot{x} = f(x) \text{ with I.C. } x(0) = x_0 \tag{2}$$

- Assumptions: (i) f Lipschitz continuous; (ii) origin is an isolated equilibrium $f(0) = 0$
- Stability Definitions: The equilibrium $x = 0$ is called
  - **stable** in the sense of Lyapunov, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|x(0)\| \le \delta \Rightarrow \|x(t)\| \le \epsilon, \forall t \ge 0$$

Lyapunov Stability Definitions (2/2)

- **asymptotically stable** if it is stable and δ can be chosen so that

$$\|x(0)\| \leq \delta \Rightarrow x(t) \to 0 \text{ as } t \to \infty$$

- **exponentially stable** if there exist positive constants δ, λ, c such that

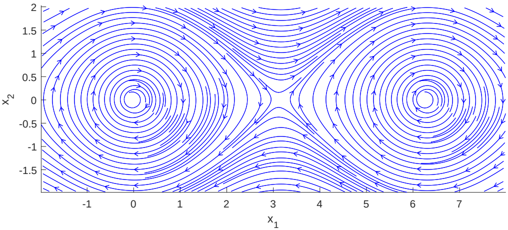
$$\|x(t)\| \leq c\|x(0)\|e^{-\lambda t}, \quad \forall \|x(0)\| \leq \delta$$

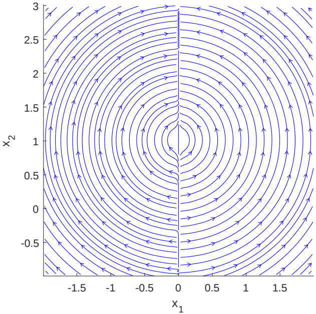
- **globally asymptotically/exponentially stable** if the above conditions holds for all $\delta > 0$

- Region of Attraction: $R_A \triangleq \{x \in \mathbb{R}^n : \text{ whenever } x(0) = x, \text{ then } x(t) \to 0\}$

Stability Examples using 2D Phase Portrait (1/2)

- Undamped pendulum with no driving





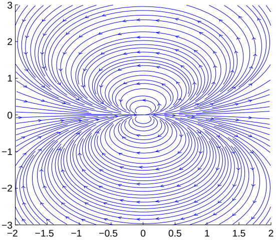
- Closed-loop dynamics under adaptive control:

$$\begin{cases} \dot{y} = y + u \\ u = -ky, \dot{k} = y^2 \end{cases}$$

Stability Examples using 2D Phase Portrait (2/2)

Does attractiveness implies stable in Lyapunov sense?

- Answer is NO. e.g.: $\begin{cases} \dot{x}_1 = x_1^2 - x_2^2 \\ \dot{x}_2 = 2x_1 x_2 \end{cases}$



- By inspection of its vector field, we see that $x(t) \to 0$ for all $x(0) \in \mathbb{R}^2$

- However, there is no δ-ball satisfying the Lyapunov stability condition

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How to verify stability of a system? (1/2)

- Find explicit solution of the ODE $x(t)$ and check stability definitions
  - typically not possible for nonlinear systems

- Numerical simulations of ODE do not provide stability guarantees and offer limited insights

- Need to determine stability without explicitly solving the ODE

- Preferably, analysis only depends on the vector field

How to verify stability of a system? (2/2)

- The most powerful tool is: *Lyapunov function*

- State trajectory $x(t)$ governed by complex dynamics in $\mathbb{R}^n$

- Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ maps $x(t)$ to a scalar function of time $V(x(t))$

- If the function is designed such that: $[x(t) \to \text{equilibrium}] \Leftrightarrow [V(x(t)) \to 0]$. Then we can study $V(x(t))$ as function of time t to infer stability of the state trajectory in $\mathbb{R}^n$.

Sign Definite Functions

Assume that $0 \in D \subseteq \mathbb{R}^n$

- $g : D \to \mathbb{R}$ is called positive semidefinite (PSD) on D if $g(0) = 0$ and $g(x) \geq 0, \forall x \in D$
  - For quadratic function: $g(x) = x^T P x$: [g is PSD] $\Leftrightarrow$ [P is a PSD matrix]

- $g : D \to \mathbb{R}$ is called positive definite (PD) on D if $g(0) = 0$ and $g(x) > 0, \forall x \in D \setminus \{0\}$
  - Similarly, if $g(x) = x^T P x$ is quadratic, then [g is PD] $\Leftrightarrow$ [P is a PD matrix]

- g is negative semidefinite (NSD) if $-g$ is PSD

- $g : \mathbb{R}^n \to \mathbb{R}$ is radically unbounded if $g(x) \to \infty$ as $\|x\| \to \infty$

Lyapunov Stability Theorem

[Lyapunov Theorem]: Let $D \subset \mathbb{R}^n$ be a set containing an open neighborhood of the origin. If there exists a $\mathcal{C}^1$ function $V : D \to \mathbb{R}$ such that

$$V \text{ is PD} \tag{3}$$

$$\dot{V}(x) \triangleq \nabla V(x)^T f(x) \text{ is NSD} \tag{4}$$

then the origin is stable. If in addition,

$$\dot{V}(x) \triangleq \nabla V(x)^T f(x) \text{ is ND} \tag{5}$$

then the origin is asymptotically stable.

Remarks:

- A PD $\mathcal{C}^1$ function satisfying (4) or (5) will be called a **Lyapunov function**

- Under condition (5), if V is also radially unbounded
  $\Rightarrow$ globally asymptotically stable

Proof of Lyapunov Stability Theorem (1/3)

Main idea:

Proof of Lyapunov Stability Theorem (2/3)

Sketch of proof of Lyapunov stability theorem:

- First show stability under condition (4):
  - Define sublevel set: $\Omega_b = \{x \in \mathbb{R}^n : V(x) \leq b\}$. Condition (4) implies $V(x(t))$ nonincreasing along system trajectory $\Rightarrow$ If $x(0) \in \Omega_b$, then $x(t) \in \Omega_b$, $\forall t$.
  - Given arbitrary $\epsilon > 0$, if we can find δ, b such that $B(0,\delta) \subseteq \Omega_b \subseteq B(0,\epsilon)$. Then the Lyapunov stability conditions are satisfied. Below is to show how we can find such b and δ.
  - V is continuous $\Rightarrow$ $m = \min_{\|x\|=\epsilon} V(x)$ exists (due to Weierstrass theorem). In addition, V is PD $\Rightarrow$ $m > 0$. Therefore, if we choose $b \in (0, m)$, then $\Omega_b \subseteq B(0,\epsilon)$.
  - $V(x)$ is continuous at origin $\Rightarrow$ for any $b > 0$, there exists $\delta > 0$ such that $|V(x) - V(0)| = V(x) < b, \forall x \in B(0,\delta)$. This implies that $B(0,\delta) \subseteq \Omega_b$.

Proof of Lyapunov Stability Theorem (3/3)

- Second, show asymptotic stability under condition (5):
  - We know $V(x(t))$ decreases monotonically as $t \to \infty$ and $V(x(t)) \geq 0$, $\forall t$. Therefore, $c = \lim_{t\to\infty} V(x(t))$ exists. So it suffices to show $c = 0$. Let us use a contradiction argument.
  - Suppose $c \neq 0$. Then $c > 0$. Therefore, $x(t) \notin \Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}, \forall t$. We can choose $\beta > 0$ such that $B(0, \beta) \subseteq \Omega_c$ (due to continuitiy of V at 0).
  - Now let $a = -\max_{\beta \leq \|x\| \leq \epsilon} \dot{V}(x)$. Since V is ND, then $a > 0$
  - $V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \leq V(x(0)) - a \cdot t < 0$ for sufficiently large t. $\Rightarrow$ contradiction!

Exponential Lyapunov Function

Definition 3 (Exponential Lyapunov Function).

$V : D \to \mathbb{R}$ is called an Exponential Lyapunov Function (ELF) on $D \subset \mathbb{R}^n$ if $\exists k_1, k_2, k_3, \alpha > 0$ such that

$$\begin{cases} k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \\ \mathcal{L}_f V(x) \leq -k_3 V(x) \end{cases}$$

Theorem 1 (ELF Theorem).

If system (2) has an ELF, then it is exponentially stable.

Stability Analysis Examples (1/2)

Example 1.

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 = x_1 - x_2 - x_1^2 - x_2^3 \end{cases} \qquad \text{Try } V(x) = \|x\|^2$$

Stability Analysis Examples (2/2)

Example 2.

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 \end{cases}$$

- Can we find a simple quadratic Lyapunov function? First try: $V(x) = x_1^2 + x_2^2$

- In fact, the system does not have any (global) polynomial Lyapunov function. But it is GAS with a Lyapunov function $V(x) = \ln(1 + x_1^2) + x_2^2$.

Outline

Stability of Linear Systems

Consider autonomous linear system: $\dot{x} = f(x) = Ax$.

- Recall solution to the linear system: $x(t) = e^{At}x(0)$
- Only possible equilibrium is origin $x = 0$
- Fact: Origin asympt. stable $\quad \Leftrightarrow \quad Re(\lambda_i) < 0$ for all eigenvalues λ_i of A
- Discrete time system: $x(k+1) = Ax(k)$ is asymp. stable iff eig(A) inside unit circle

Lyapunov Function of Linear Systems

- Consider a quadratic Lyapunov function candidate: $V(x) = x^T P x$, with $P \in \mathbb{R}^{n \times n}$
  - V is PD $\Rightarrow P \succ 0$
  - $\mathcal{L}_f V$ is ND $\Rightarrow$

Stability Conditions for Linear Systems

Theorem 2 (Stability Conditions for Linear System).

For an autonomous Linear system $\dot{x} = Ax$. The following statements are equivalent.

- *System is (globally) asymptotically stable*
- *System is (globally) exponentially stable*
- $Re(\lambda_i) < 0$ *for all eigenvalues* λ_i *of* A
- *System has a quadratic Lyapunov function*
- *For any symmetric* $Q \succ 0$*, there exists a symmetric* $P \succ 0$ *that solves the following Lyapunov equation:*

$$PA + A^T P = -Q$$

and $V(x) = x^T P x$ *is a Lyapunov function of the system.*

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When There is a Lyapunov Function?

- Converse Lyapunov Theorem for Asymptotic Stability

$$\begin{cases} \text{origin asymptotically stable;} \\ f \text{ is locally Lipschitz on } D \\ \text{with region of attraction } R_A \end{cases} \Rightarrow \exists V \text{ s.t.} \begin{cases} \text{V is continuuos and PD on } R_A \\ \mathcal{L}_f V \text{ is ND on } R_A \\ V(x) \to \infty \text{ as } x \to \partial R_A \end{cases}$$

- Converse Lyapunov Theorem for Exponential Stability

$$\begin{cases} \text{origin exponentially stable on } D; \\ f \text{ is } \mathcal{C}^1 \end{cases} \Rightarrow \exists \text{ an ELF } V \text{ on } D$$

- Proofs are involved especially for the converse theorem for asymptotic stability

- **IMPORTANT**: proofs of converse theorems often assume the knowledge of system solution and hence are not constructive.

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What about Discrete Time Systems?

- So far, all our definitions, results, examples are given using continuous time dynamical system models.
- All of them have discrete-time counterparts. The ideas and conclusions are the "same" (in spirit)
- For example, given autonomous discrete-time system: $x(k+1) = f(x(k))$ with $f(0) = 0$ (origin is an equilibrium).
  - Rate of change of a function $V(x)$ along system trajectory can be defined as:

$$\Delta_f V(x) \triangleq V(f(x)) - V(x)$$

  - Asymptotically stable requires:

$$V \text{ is PD} \quad \text{and} \quad \Delta_f V \text{ is ND}$$

  - Exponentially stable requires:

$$k_1 \|x\|^\alpha \leq V(x) \leq k_2 \|x\|^\alpha \quad \text{and} \quad \Delta_f V(x) \leq -k_3 V(x)$$

  - ....

Concluding Remarks

- We have learned different notions of internal stability, e.g. stability in Lyapunov sense, asymptotic stability, globally asymptotic stability (G.A.S), exponential stability, globally exponential stability (G.E.S)
- Sufficient condition to ensure stability is often the existence of a properly defined Lyapunov function
- Key requirements for a Lyapunov function:
  - positive definite and is zero at the system equilibrium
  - decrease along system trajectory
- For linear system: G.A.S $\Leftrightarrow$ G.E.S $\Leftrightarrow$ Existence of a quadratic Lyapunov function
- The definitions and results in this lecture have sometimes been stated in simplified forms to facilitate presentation. More general versions can be found in standard textbooks on nonlinear systems
- **Next Lecture**: Semidefinite Programming and computational stability analysis

More Discussions

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