MEE5114 Advanced Control for Robotics

Lecture 12: Semidefinite Programming for Stability Analysis

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Outline

- Linear Matrix Inequalities
- Semidefinite Programming Problems
- S-Procedure
- Some Examples
- Conclusion

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Linear Matrix Inequalities (1/4)

- *Standard form*: Given symmetric matrices $F_0, \ldots, F_m \in \mathcal{S}^n$,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0$$

  is called a *Linear Matrix Inequality* in $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$

- The function $F(x)$ is affine in x

- The constraint set $\{x \in \mathbb{R}^n : F(x) \succeq 0\}$ is nonlinear but convex

Linear Matrix Inequalities (2/4)

Example 1 (LMI in Standard Form).

Characterize the constraint set: $F(x) = \begin{bmatrix} x_1 + x_2 & x_2 + 1 \\ x_2 + 1 & x_3 \end{bmatrix} \succeq 0$

Linear Matrix Inequalities (3/4)

- General Linear Matrix Inequalities (LMI)
  - Let $\mathcal{X}$ be a finite-dimensional real vector space.
  - $F : \mathcal{X} \to \mathcal{S}^n$ is an *affine* mapping from $\mathcal{X}$ to $n \times n$ symmetric matrices
  - Then $F(X) \succeq 0$ is called also an LMI in variable $X \in \mathcal{X}$
  - Translation to standard form: Choose a basis $X_1, \ldots, X_m$ of $\mathcal{X}$ and represent $X = x_1 X_1 + \cdots + x_m X_m$ for any $X \in \mathcal{X}$. For a given affine mapping $F : \mathcal{X} \to \mathcal{S}^n$, we can define $\hat{F} : \mathbb{R}^m \to \mathcal{S}^n$ as

$$\hat{F}(x) \triangleq F(X) = F(0) + \sum_{i=1}^{m} x_i [F(X_i) - F(0)]$$

where x is the coordinate of X w.r.t. the basis $X_1, \ldots, X_m$.

Linear Matrix Inequalities (4/4)

Example 2.

Find conditions on matrix P to ensure that $V(x) = x^T P x$ is a Lyapunov function for a linear system $\dot{x} = Ax$

Schur Complement Lemma (1/2)

Lemma 1 (Schur Complement Lemma).

Define $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. *The following three sets of inequalities are equivalent.*

$$M \succ 0 \quad \Leftrightarrow \quad \begin{cases} A \succ 0 \\ C - B^T A^{-1} B \succ 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} C \succ 0 \\ A - BC^{-1}B^T \succ 0 \end{cases}$$

- **Proof:** The lemma follows immediately from the following identities:

$$\begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$$

$$\begin{bmatrix} I & -BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & 0 \\ -C^{-1}B^T & I \end{bmatrix} = \begin{bmatrix} A - BC^{-1}B^T & 0 \\ 0 & C \end{bmatrix}$$

Schur Complement Lemma (2/2)

- The proof of Schur complement lemma also reveals more general relations between the numbers of negative, zero, positive eigenvalues of
  - M vs. A and $C - B^T A^{-1} B$
  - M vs. C and $A - BC^{-1} B^T$
- Schur complement lemma is a very useful result to transform nonlinear (quadratic or bilinear) matrix inequalities to linear ones.

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Semidefinite Programming (1/3)

- **Semidefinite Programming (SDP) Problem**: Optimization problem with linear objective, and Linear Matrix Inequality and linear equality constraints:

$$\begin{cases} \text{minimize:} & c^T x \\ \text{subject to:} & F_0 + x_1 F_1 + \cdots + x_m F_m \succeq 0 \\ & Ax = b \end{cases} \tag{1}$$

- Linear *equality* constraint in (1) can be eliminated. So essentially SDP can be viewed as optimizing linear function subject to only LMI constraints.

- SDP is a particular class of convex optimization problem. Global optimal solution can be found efficiently.

- Optimizing nonlinear but convex cost function subject to LMI constraints is also a convex optimization that can often be solved efficiently.

Semidefinite Programming (2/3)

Standard forms of SDP in *matrix variable*:

- **SDP Standard Prime Form**:

$$\begin{cases} \min_{X \in \mathcal{S}^n} : & f_p(X) = C \bullet X \\ \text{subject to:} & A_i \bullet X = b_i, i = 1, \dots, m \\ & X \succeq 0 \end{cases} \tag{2}$$

- **SDP Dual form**:

$$\begin{cases} \max_{y \in \mathbb{R}^m} : & f_d(y) = b^T y \\ \text{subject to:} & \sum_{i=1}^m y_i A_i \preceq C \end{cases} \tag{3}$$

- One can derive the dual from the prime using either standard Lagrange duality method or more specialized Fenchel duality results
- The dual form (3) is equivalent to (1) (after eliminating the equality constraint $Ax = b$ in (1))

Semidefinite Programming (3/3)

- **SDP Weak Duality**: $f_p(X) \geq f_d(y)$ for any primal and dual feasible X and y

- **SDP Strong Duality**: $f_p(X^*) = f_d(y^*)$ holds under Slater's condition:

- Many control and optimization problem can be formulated or translated into SDP problems

- Various computationally difficult optimization problems can be effectively approximated by SDP problems (SDP relaxation...)

- We will see some examples after introducing an important technique: *S-procedure*

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S-Procedure (1/2)

- Many stability/engineering problems require to certify that a given function is sign-definite over certain subset of the space

- Mathematically, this condition can be stated as follows:

$$g_0(x) \geq 0 \quad \text{on} \quad \{x \in \mathbb{R}^n | g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \tag{4}$$

- Given functions $g_0, \ldots, g_m$, we want to know whether the condition holds. Sometimes we may also want to find a g_0 satisfying this condition for given $g_1, \ldots, g_m$.

- Conservative but useful condition: $\exists$ PSD functions $s_i(x)$ s.t.

$$g_0(x) - \sum_i s_i(x) g_i(x) \geq 0, \forall x \in \mathbb{R}^n$$

This is the so-called **Generalized S-Procedure**

S-Procedure (2/2)

Now consider an important special case: $g_i(x) = x^T G_i x, i = 0, 1, \ldots$ are quadratic functions

- Requirement (4) becomes:

$$\forall x \in \mathbb{R}^n, \quad x^T G_1 x \geq 0, \ldots, x^T G_k x \geq 0 \quad \Rightarrow \quad x^T G_0 x \geq 0$$

- Sufficient condition (S-procedure): $\exists \alpha_1, \ldots, \alpha_m \geq 0$ with

$$G_0 \succeq \alpha_1 G_1 + \cdots + \alpha_m G_m$$

- S-Procedure is lossless if $m = 1$ and $\exists \hat{x}$ s.t. $\hat{x}^T G_1 \hat{x} > 0$ (constraint qualification)

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Some Examples (1/4)

Example 3 (Eigenvalue Optimization).

Given symmetric matrices $A_0, A_1, \ldots, A_m$. Let $S(w) = A_0 + \sum_i w_i A_i$. Find weights $\{w_i\}_{i=1}^m$ to minimize $\lambda_{\max}(S(w))$

Some Examples (2/4)

Example 4 (Ellipsoid inequality).

Given $R \in \mathcal{S}^n_{++}$, the set $E = \{x \in \mathbb{R}^n : (x - x_c)^T R(x - x_c) < 1\}$ is an ellipsoid with center x_c. Find the point in E that is the closet to the origin.

Some Examples (3/4)

Example 5 (Linear Feedback Control Gain Design).

Given a linear control system $\dot{x} = Ax + Bu$ with linear state feedback $u = Kx$. Find K to stabilize the system

Some Examples (4/4)

Example 6 (Robust Stabiltiy).

Given system $\dot{x} = Ax + u$ with uncertain feedback $u = g(x)$. Suppose all we know is that the feedback law satisfies: $\|g(x)\|^2 \leq \beta\|x\|^2$. Find Lyapunov function $V(x) = x^T P x$ to ensure exponential stability.

Concluding Remarks

- Linear matrix inequalities impose convex constraints

- Semidefinite programming problem: optimize linear cost subject to LMI constraints

- SDP has broad applications in various engineering fields: signal processing, networking, communication, control, machine learning, big data...

References

More Discussions

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