1. Given a linear system $\dot{x} = Ax$ and a quadratic function $V(x) = x^T Px$, where P is an $n \times n$ symmetric matrix. Derive the conditions for P under which V will be a Lyapunov function for exponential stability that satisfies $\|x(t)\|^2 \leq \beta c^t \|x(0)\|^2$, where $c \in (0, 1)$.

2. Show that the system $\dot{x} = f(x) = \begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 \end{cases}$ is globally asymptotically stable (hint: try $V(x) = \ln(1 + x_1^2) + x_2^2$ as a Lyapunov function)

3. Consider a discrete time system $x(k+1) = Ax(k) + Bu(k)$, with linear feedback law $u(k) = -Kx(k)$. Write down the closed-loop dynamics, and derive conditions for $V(x) = x^T Px$ to be discrete time Lyapunov function for asymptotic closed-loop stability.

4. Show that the PSD cone is acute, i.e., $\forall A, B \in \mathcal{S}_+^n$, we have $tr(AB) \geq 0$. (Hint: decompose A using unitary matrix Q, i.e. $A = Q\Lambda Q^T$, and then use the same Q to define another matrix $C = QBQ^T$. The trace $tr(AB)$ can be computed directly in terms of the entries in C and Λ)

5. Given a symmetric matrix $A \in \mathcal{S}^n$, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the smallest and largest eigenvalues of A. Show that

$$\begin{cases} \lambda_{\min}(A) \geq \mu \\ \lambda_{\max}(A) \leq \beta \end{cases} \Leftrightarrow \mu I \preceq A \preceq \beta I$$

6. Suppose $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$ are convex. Show that the pointwise maximum function $f(x) = \max\{f_1(x), f_2(x)\}$ is also convex.