MEE5114 Spring 2022 **Homework #6**
Assigned May 10, 2022 Due: May 17, 2022

- Sumbit your homework (pdf) through Blackboard
- To receive credits, please write down all the necessary steps leading to final answer.

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1. **Schur complement lemma**: Find an equivalent semidefinite condition of the form $G(x) \succeq 0$ for each of the following statements. Make sure the matrix $G(x)$ you obtained is affine w.r.t. x, where x is a vector or matrix variable of appropriate dimension. Please show your steps.
   (a) (Singular value bound): $\sigma(A(x)) < \beta$, where $A : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ is affine in $x \in \mathbb{R}^n$ and $\sigma(\cdot)$ denotes the singular value of a matrix.
   (b) (Riccati inequality): $A^T x + xA + xBR^{-1}B^T x + Q \prec 0$, with $x \in \mathcal{S}^n_{++}$, $R \in \mathcal{S}^p_{++}$, $Q \in \mathcal{S}^n$ and $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times p}$.

2. **Ellipsoid:** Ellipsoid in $\mathbb{R}^n$ have two equivalent representations: (i) $E_1(P, x_c) = \{x \in \mathbb{R}^n : (x-x_c)P^{-1}(x-x_c) \leq 1\}$ and (ii) $E_2(A, x_c) = \{Au+x_c : \|u\|^2 \leq 1\}$. The second representation can be derived from the first by letting $A = P^{1/2}$. Given $E_1(P, x_c)$ with $P \in \mathcal{S}^n_{++}$, its volume is $\nu_n\sqrt{det(P)}$ where ν_n is the volume of unit ball in $\mathbb{R}^n$, its semi-axes directions are given by the eigenvectors of P and the lengths of semi-axes are $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P.
   (a) Given a half space $\{x \in \mathbb{R}^n : a^T x \leq 1\}$. Show that the Ellipsoid $E_2(A, 0)$ is contained in the half space if and only if $a^T AA^T a \leq 1$.
   (b) Note that for any $P \in \mathcal{S}^n_{++}$, the function $log(det(P))$ is concave in the matrix variable P. Formulate a convex optimization problem to find the matrix $P \in \mathcal{S}^n_{++}$ such that $E_1(P, 0)$ is the largest ellipsoid contained in the polyhedron $\{x \in \mathbb{R}^n : a_i^T x \leq 1, i = 1, \ldots, m\}$
   (c) Use Drake to solve the above problem with $a_1^T = [-1, 1], a_2^T = [2, -1], a_3^T = [1, 3], a_4^T = [-2, -5]$. Visualize the polyhedron region and your ellipsoid solution (you can use Matlab for the visualization if you prefer matlab)

3. **Stability of Lur'e system:** Consider a nonlinear system
$$\dot{x} = Ax + b\phi(t, c^T x)$$
   where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and for each time t, the function $\phi(t, \cdot) : \mathbb{R} \to \mathbb{R}$ satisfies sector nonlinearity $|\phi(t, y)| \leq \alpha|y|$ for all y (but is otherwise unknown). Such a system represents a very general class of control systems involving time-varying nonlinearities and/or uncertainties, and is often called a *Lur'e* problem.

   We would like to find a positive definite Lyapunov function $V(x) = x^T Px$ that satisfies $\dot{V}(x) \leq -\beta V(x)$ for all x, and for any function ϕ satisfying the inequality given above. You can assume that A, b, c, α, β are given.
   (a) Explain how to find such a P (or determine that no such P exists) by expressing the problem as an LMI.
   (b) Use CVX to construct such a Lyapunov function for the following instance of Lur'e problem:
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha = 0.7, \beta = 0.1$$

4. **Stabilization via LMIs:** Consider the time-varying LDS (linear dynamical system)

$$\dot{x}(t) = A(t)x(t) + Bu(t),$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, where $A(t) \in \{A_1, \ldots, A_M\}$. Thus, the dynamics matrix $A(t)$ can take any of M values, at any time. We seek a linear state feedback gain matrix $K \in \mathbb{R}^{m \times n}$ for which the closed-loop system

$$\dot{x}(t) = [A(t) + BK]x(t),$$

is globally asymptotically stable. But even if you are given a specific state feedback gain matrix K, this is very hard to determine. So we will require the existence of a quadratic Lyapunov function that establishes exponential stability of the closed-loop system, *i.e.*, a matrix $P = P^T \succ 0$ for which

$$\dot{V}(z,t) = z^T[(A(t) + BK)^T P + P(A(t) + BK)]z \leq -\beta V(z)$$

for all z, and for any possible value of $A(t)$. (The parameter $\beta > 0$ is given, and sets a minimum decay rate for the closed-loop trajectories.)

So roughly speaking we seek

- a stabilizing state feedback gain, and
- a quadratic Lyapunov function that certifies the closed-loop performance.

In this problem, you will use LMIs to find both K and P, *simultaneously.*

(a) Pose the problem of finding P and K as an LMI problem. *Hint:* Starting from the inequality above, you will not get an LMI in the variables P and K (although you will have a set of matrix inequalities that are affine in K, for fixed P, and linear in P, for fixed K). Use the new variables $X = P^{-1}$ and $Y = KP^{-1}$. Be sure to explain why you can change variables.

(b) Carry out your method for the specific problem instance

$$A_1 = \begin{bmatrix} -0.5 & 0.3 & 0.4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.7 & 0.1 & -0.2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.6 & -0.7 & 0.2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$B = (1, 0, 0)$, and $\beta = 1$. (Thus, we require a closed-loop decay at least as fast as $e^{-t/2}$.)

5. **Derivation of Dual SDP (Optional; will not be graded):** Consider the prime and dual SDP problems discussed in class.

$$(\textbf{PSDP}): \begin{cases} \min\limits_{X} & C \bullet X \\ \text{subj.:} & A_i \bullet X = b_i, \\ & i = 1, \ldots, m \\ & X \succeq 0 \end{cases} \qquad (\textbf{DSDP}): \begin{cases} \max\limits_{y \in \mathbb{R}^m} & b^T y \\ \text{subj.:} & \sum_{i=1}^m y_i A_i \preceq C \end{cases}$$

This homework problem intends to guide you through the derivation of the dual SDP problem. We need to introduce a few definitions. In general, the Lagrangian dual of a constrained optimization problem $\min_{x\in\mathcal{X}}\{f(x) : h(x) = 0\}$ is defined as

$$\max_{y\in\mathbb{R}^m}\left(\min_{x\in\mathcal{X}}\{f(x) + y^T h(x)\}\right)$$

To derive the desired dual form, we also need a concept: *dual cone*. Let $\mathcal{V}$ be an inner product space. Suppose $K \subseteq \mathcal{V}$ is a cone, i.e., $\lambda x \in \mathcal{K}$ for all $x \in \mathcal{K}$. The dual cone of K is defined as

$$Dual(K) = \{z \in \mathcal{V} : \langle z, x\rangle \geq 0, \forall x \in \mathcal{K}\}$$

The dual of the PSD cone is thus $dual(\mathcal{S}^n_+) = \{Y \in \mathcal{S}^n : Y \bullet X \geq 0, \forall X \in \mathcal{S}^n_+\}$. A well-known fact about PSD cone is that it is self-dual, i.e. $\mathcal{S}^n_+ = dual(\mathcal{S}^n_+)$.

(a) Write down the Lagrangian dual problem for (PSDP).

(b) Show that $\min_{X\succeq 0}\langle Y, X\rangle = \begin{cases} 0 & \text{if } Y \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$ (Hint: use the fact that $\mathcal{S}^n_+$ is acute and self-dual.

(c) Show that the Lagrangian dual problem of (PSDP) can be reduced to the form (DSDP) given above.