ME424 Fall 2022 | **Homework # 3**

Assigned September 26, 2022 | Due: October 9 (Sunday), 2022

- Please submit your solution through Blackboard system
- To receive credits, please write down all the necessary steps leading to final answer.

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1. **Sensor Networks:** Consider a sensor network with 5 nodes as shown in Fig. 1. At each discrete time step k, each sensor i, $i = 1, \ldots, 5$, sends its value $x_i(k)$ to other sensors that are directly connected with sensor i. For example, sensor 3 sends its value $x_3(k)$ to sensor 2, while it receives data from sensor 1 and sensor 4. Find the discrete-time state space model for the dynamics of the sensor values under the following two cases

   (a) For each sensor i, the new value at time $k+1$, $x_i(k+1)$, is the maximum among the data it receives from the neighbors and its own value at time k.

   (b) For each sensor i, the new value at time $k+1$, $x_i(k+1)$, is the average of the data it receives from the neighbors and its own value at time k.

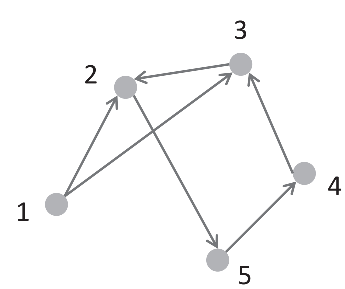

Figure 1: A small sensor network

2. **Power Electronics:** Consider the buck converter as show in Fig. 2 with ideal switches. Let the inductor current and capacitor voltage be the system states: $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$. Let the output of the system be $y(t) = v_o(t)$. Find a linear state space model for each switching position as shown in Fig. 2.

3. **Permanent Magnet DC Motor Model:** Please read the supplemental material associated with HW3 (Section 8.2 and 8.3). Derive the state-space model for the motor + load system shown in Figure 8.5 or equivalently Fig. 8.6 with the armature voltage V_a as control input and rotor position θ_m as the output. Note that this is a

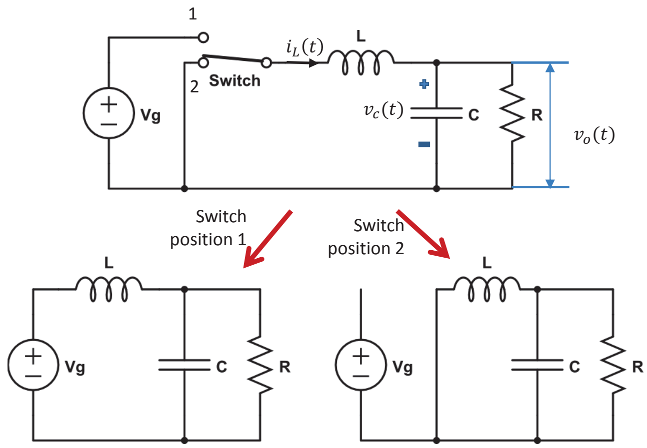

Figure 2: Buck converter

linear system with the load torque as disturbance. You should obtain a model of the following form:

$$\begin{cases} \dot{x} = Ax + Bu + Fd \\ y = Cx + Du \end{cases}$$

where the disturbance $d = \tau_l$ is the load torque.

4. **Linearized Pendulum Model:** Consider a pendulum as shown in Fig. 3 with linear friction at the pivot. Assume the rod is rigid with zero mass and its length is l. The equations of motion is given by

$$ml^2\ddot{\theta} = -mgl\sin\theta - bl^2\dot{\theta}$$

where b is the friction coefficient.

   (a) Find an equivalent continuous time state-space model. (choose your state vector as $x = [\theta, \dot{\theta}]^T$, and your output as $y = \theta$. There is no control input in this case)

   (b) Compute the linearized model around $\hat{x} = [0, 0]^T$.

   (c) Find the discrete time linear state space model with sampling period $\Delta t = 2$ (sec).

5. **Least Squares (Curve fitting):** Suppose that you want to fit a function $y(t) = \frac{3t^2}{1+t^3}$ with a 2nd order polynomial over the interval $t \in [0, 1]$. This can be formulated as a least-squares problem: Find a parameter vector $\hat{\theta} = [\alpha_0, \alpha_1, \alpha_2]$ to minimize

$$J(\hat{\theta}) \triangleq \sum_{i=1}^{N} \left[y(t_i) - (\alpha_0 + t_i + \alpha_1 t_i^2 + \alpha_2 t_i^3)\right]^2$$

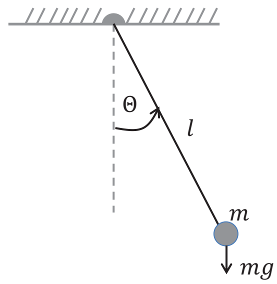

Figure 3: Pendulum Example

for some observation points $t_1, t_2, \ldots, t_N \in [0, 1]$.

(a) Suppose you choose 5 observation points $t_i = 0.2 \cdot i$, $i = 1, \ldots, 5$. Write down the H matrix, and y vector in the standard least-squares formulation.

(b) Suppose you choose 10 observation points $t_i = 0.1 \cdot i$, $i = 1, \ldots, 10$. Use Python to compute the least-squares estimate of $\hat{\theta}$. Plot both $y(t)$ and $\hat{y}(t) = (\hat{\alpha}_0 + t + \hat{\alpha}_1 t^2 + \hat{\alpha}_2 t^3)$ for $t = 0, 0.01, 0.02, \ldots 1$ on the same plot.

6. **Least Squares (Discrete Fourier Transform (DFT)):** DFT can be viewed as approximating a (complex-valued) discrete-time signal using a weighted sum of sinusoidal signals. With complex data, the least-squares solution is given by:

$$\theta_{LS} = (H^* H)^{-1} H^* y \tag{1}$$

where $\cdot^*$ means conjugate transpose (i.e. taking elementwise complex conjugate of the transpose). For example $[5, 1 + 2j]^* = \begin{bmatrix} 5 \\ 1 - 2j \end{bmatrix}$.

For N-point DFT, we aim to represent a signal $x[n]$ using N Fourier basis waveforms

$$x[n] \approx \sum_{k=0}^{N-1} X_k \cdot \phi_k[n], \quad n = 0, \ldots, N-1$$

where the kth ($k = 0, \ldots, N-1$) Fourier basis is given by

$$\phi_k[n] = \frac{1}{N} \exp\left(\frac{j2\pi kn}{N}\right), n = 0, \ldots, N-1.$$

Let $\hat{\theta} = \begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix}$. Define the signal representation error as

$$J(\hat{\theta}) = \sum_{n=0}^{N-1} \left\{ \left(x[n] - \sum_{k=0}^{N-1} X_k \cdot \phi_k[n] \right)^2 \right\}$$

(a) Write $J(\hat{\theta})$ in standard least-squares form, i.e., find the vector y and matrix H so that $J(\hat{\theta}) = \|y - H\hat{\theta}\|^2$. Your answer for y, H should be in terms of $x[n]$ and $\phi_k[n]$, where $k, n = 0, \ldots, N-1$.

(b) By trigonometric identifies, it can be easily verified that the Fourier basis are orthogonal, i.e.,

$$\phi_k^* \phi_m = \sum_{n=0}^{N-1} \phi_k^*[n]\phi_m[n] = \sum_{n=0}^{N-1} \frac{1}{N} \exp\left(\frac{-j2\pi kn}{N}\right) \cdot \frac{1}{N} \exp\left(\frac{j2\pi mn}{N}\right) = \begin{cases} 0 & \text{if } k \neq m \\ \frac{1}{N} & \text{if } k = m \end{cases}$$

Given the above fact and the complex least-squares formula in (1), derive an analytical expression for $\hat{X}_k$, $k = 0, \ldots, N-1$, wchih is the least-squares estimate of X_k (i.e. the kth element of $\hat{\theta}_{LS}$). You should provide all the derivation steps. Hint: Using the orthogonal condition given above, you should be able to show that H^*H (and hence $(H^*H)^{-1}$) is a diagonal matrix. With that, you can easily derive a formula for $\hat{X}_k$

(c) Let $x[n] = \sin[(1/5\pi)n]$, $n = 0, \ldots, 30$. Write a simple Python code to compute the DFT using the above least square approach. Your code needs to compute the H matrix, y vector, and the resulting least square solution. Compare your answer with the solution given by the "scipy.fft" module in Python.