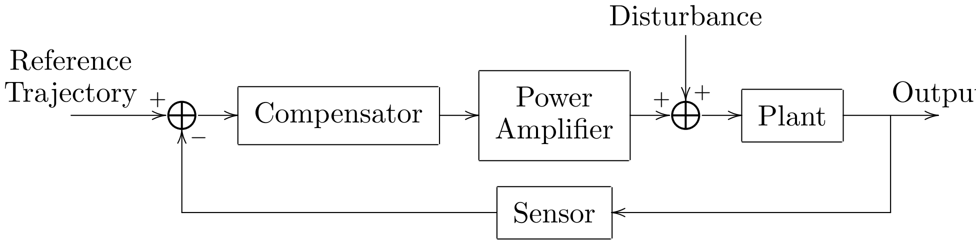

Figure 8.1: Basic structure of a feedback control system. The compensator measures the **error** between a **reference** and a measured **output** and produces signals to the plant that are designed to drive the error to zero despite the presence of disturbances.

8.2 Actuator Dynamics

Robot manipulators are equipped with actuators to move the joints through programmed motion trajectories in order to complete given tasks. These actuators may be electric, hydraulic, or pneumatic. In this section we restrict our attention to the dynamics of **permanent magnet DC motors**, as these are the simplest actuators to analyze and are commonly used in robot manipulators. Other types of electric motors, in particular **AC motors** and so-called **brushless DC motors**, are also used as actuators for robots but we will not discuss their dynamics here.

A DC motor works on the principle that a current-carrying conductor in a magnetic field experiences a force $F = i \times \phi$, where ϕ is the magnetic flux, i is the current in the conductor, and $\times$ is the vector cross product. The motor itself consists of a fixed **stator** and a movable **rotor** that rotates inside the stator as shown in Figure 8.2.

If the stator produces a radial magnetic flux ϕ and the current in the rotor (also called the **armature**) is i, then there will be a torque on the rotor causing it to rotate. The magnitude of this torque is

$$\tau_m = K_1 \phi i_a \tag{8.1}$$

where τ_m is the motor torque (Newton-meters), ϕ is the magnetic flux (webers), i_a is the armature current (amperes), and K_1 is a physical constant.

In addition, whenever a conductor moves in a magnetic field, a voltage V_b is generated across its terminals that is proportional to the velocity of the conductor in the field. This voltage, called the **back emf**, will tend to oppose the current flow in the conductor. Thus, in addition to the torque

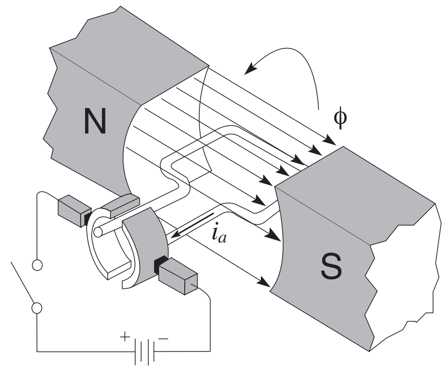

Figure 8.2: Principle of operation of a permanent magnet DC motor. The magnitude of the force (or torque) on the armature is proportional to the product of the current and magnetic flux. A **commutator** is required to periodically switch the direction of the current through the armature to keep it rotating in the same direction.

τ_m in Equation (8.1), we have the back emf relation

$$V_b = K_2 \phi \omega_m \tag{8.2}$$

where V_b denotes the back emf (volts), ω_m is the angular velocity of the rotor (radians per second), and K_2 is a proportionality constant.

DC motors can be classified according to the way in which the magnetic field is produced and the armature is designed. Here we discuss only the so-called **permanent magnet** motors whose stator consists of a permanent magnet. In this case we can take the flux ϕ to be a constant. The torque on the rotor is then controlled by controlling the armature current i_a.

Consider the schematic diagram of Figure 8.3, where

$$\begin{aligned}
V &= \text{armature voltage} \\
L &= \text{armature inductance} \\
R &= \text{armature resistance} \\
V_b &= \text{back emf} \\
i_a &= \text{armature current} \\
\theta_m &= \text{rotor position} \\
\tau_m &= \text{generated torque} \\
\tau_\ell &= \text{load torque} \\
\phi &= \text{magnetic flux due to the stator}
\end{aligned}$$

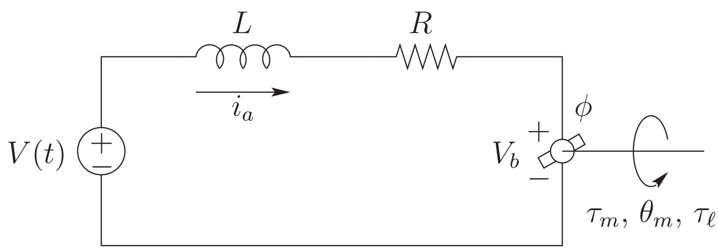

Figure 8.3: Circuit diagram for an armature controlled DC motor. The rotor windings have an effective inductance L and effective resistance R. The applied voltage V is the control input.

The differential equation for the armature current is then

$$L\frac{di_a}{dt} + Ri_a \quad = \quad V - V_b \tag{8.3}$$

Since the flux ϕ is constant, the torque developed by the motor is

$$\tau_m \quad = \quad K_1\phi i_a = K_m i_a \tag{8.4}$$

where K_m is the torque constant in N-m/amp. Also, from Equation (8.2) we have

$$V_b \quad = \quad K_2\phi\omega_m = K_b\omega_m = K_b\frac{d\theta_m}{dt} \tag{8.5}$$

where K_b is the back emf constant. It can be shown that the numerical values of K_m and K_b are the same provided MKS units[1] are used.

The torque constant can be determined from a set of torque-speed curves as shown in Figure 8.4 for various values of the applied voltage V.

When the motor is stalled, the blocked-rotor torque at the rated voltage V_r is denoted by τ_0. Using Equations (8.3) and (8.4) with $V_b = 0$ and $di_a/dt = 0$ we have

$$V_r = Ri_a = \frac{R\tau_0}{K_m} \tag{8.6}$$

Therefore the torque constant is

$$K_m \quad = \quad \frac{R\tau_0}{V_r} \tag{8.7}$$

[1]MKS units are based on the meter, kilogram, and second.

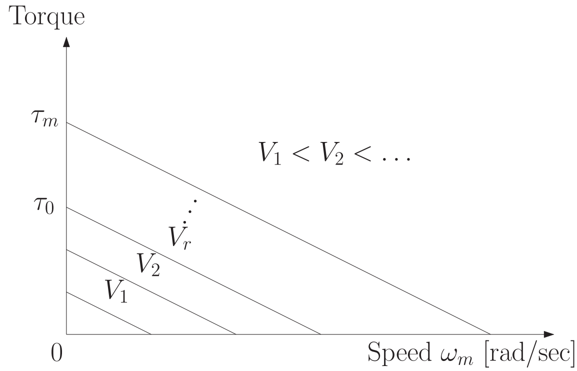

Figure 8.4: Typical torque-speed curves of a DC motor. Each line represents the torque versus speed for a given value of the applied voltage.

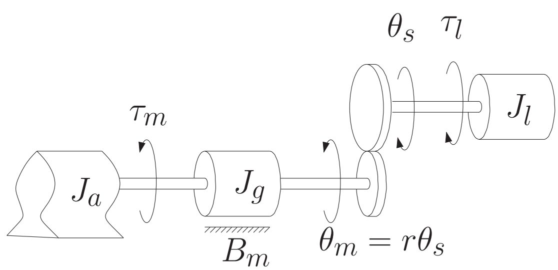

Figure 8.5: Lumped model of a single link with actuator and gear train. J_a, J_g, and J_ℓ are, respectively, the actuator, gear, and load inertias. B_m is the coefficient of motor friction and includes friction in the brushes and gears. The gear ratio is $r : 1$ with $r \gg 1$.

8.3 Load Dynamics

In this section we consider the dynamics of the DC motor in series with a gear train and load as shown in Figure 8.5. The gear ratio is $r : 1$, where r typically has values in the range 20 to 200 or more. The load is represented by the rotational inertia J_ℓ. Referring to Figure 8.5, we set $J_m = J_a + J_g$, the sum of the actuator and gear inertias.

In terms of the motor angle θ_m, the equation of motion of this system is then

$$\begin{aligned} J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} &= \tau_m - \tau_\ell / r \\ &= K_m i_a - \tau_\ell / r \end{aligned} \tag{8.8}$$

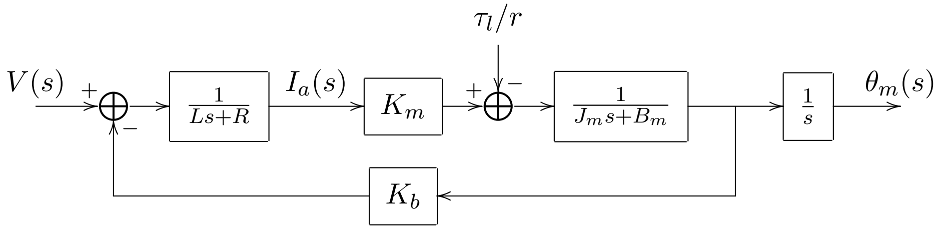

Figure 8.6: Block diagram for a DC motor system. The block diagram represents a third-order system from input voltage $V(s)$ to output position $\theta_m(s)$.

the latter equality coming from Equation (8.4). In the Laplace domain the three Equations (8.3), (8.5), and (8.8) may be combined and written as

$$(Ls + R)I_a(s) = V(s) - K_b s\Theta_m(s) \tag{8.9}$$

$$(J_m s^2 + B_m s)\Theta_m(s) = K_m I_a(s) - \tau_\ell(s)/r \tag{8.10}$$

The block diagram of the above system is shown in Figure 8.6.

The transfer function from $V(s)$ to $\Theta_m(s)$ is given, with $\tau_\ell = 0$, by (Problem 8–1)

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls + R)(J_m s + B_m) + K_b K_m\right]} \tag{8.11}$$

The transfer function from the load torque $\tau_\ell(s)$ to $\Theta_m(s)$ is given, with $V = 0$, by (Problem 8–1)

$$\frac{\Theta_m(s)}{\tau_\ell(s)} = \frac{-(Ls + R)/r}{s\left[(Ls + R)(J_m s + B_m) + K_b K_m\right]} \tag{8.12}$$

Notice that the magnitude of this latter transfer function, and hence the effect of the load torque on the motor angle, is reduced by the gear ratio r.

Frequently it is assumed that the "electrical time constant" L/R is much smaller than the "mechanical time constant" J_m/B_m. This is a reasonable assumption for many electromechanical systems and leads to a reduced order model of the actuator dynamics. If we divide numerator and denominator of Equations (8.11) and (8.12) by R and neglect the electrical time constant by setting L/R equal to zero, the transfer functions in Equations (8.11) and (8.12) become, respectively, (Problem 8–2)

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s(J_m s + B_m + K_b K_m/R)} \tag{8.13}$$

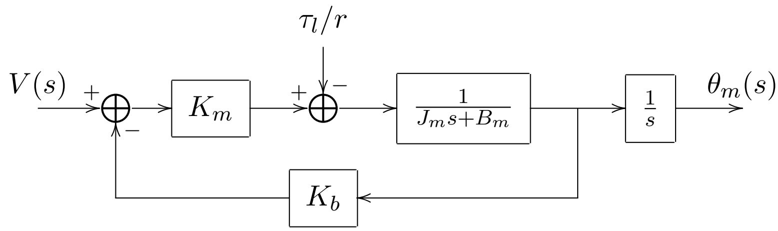

Figure 8.7: Block diagram for the reduced-order system. The block diagram now represents a second-order system.

and

$$\frac{\Theta_m(s)}{\tau_\ell(s)} = \frac{-1/r}{s(J_m(s)+B_m+K_bK_m/R)} \tag{8.14}$$

In the time domain Equations (8.13) and (8.14) represent, by superposition, the second-order differential equation

$$J_m\ddot{\theta}_m(t)+(B_m+K_bK_m/R)\dot{\theta}_m(t) = (K_m/R)V(t)-\tau_\ell(t)/r \tag{8.15}$$

The block diagram corresponding to the reduced-order system (8.15) is shown in Figure 8.7.

8.4 Independent Joint Model

In this section we refine the previous model by assuming that the load attached to the DC motor is a link of a multi-link manipulator rather than a simple rotational inertia in order to generate a more accurate description of the manipulator load dynamics. This section assumes knowledge of the Euler–Lagrange equations that we derived in Chapter 6 and may be skipped if the reader has not studied that chapter.

In Chapter 6 we obtained the following set of differential equations describing the motion of an n-degree-of-freedom manipulator (cf. Equation (6.66))

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\tau \tag{8.16}$$

We assume that the k-th component, τ_k, of the generalized force vector τ is a torque about the axis z_{k-1} if joint k is revolute, and is a force along z_{k-1} if joint k is prismatic. If the output side of the gear train is directly coupled to the joint axis, then the joint variables and motor variables are related by

$$q_k=\theta_{m_k}/r_k\ ,\ \ k=1,\ldots,n \tag{8.17}$$