Fall 2022 ME424 Modern Control and Estimation

Lecture Note 5
State-Feedback and Output Feedback Control

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Outline

- **Eigenvalues ↔ System Response**

- Full State-feedback: Eigenvalue Assignment

- Luenberger Observer Design

- Output-feedback Control and Separation Principle

- State space solution (with zero control $u(k) = 0$)
  - $x(k) = A^k x(0)$
  - **Simple Case** (Diagonalizable):
    - $A = TDT^{-1} = T\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{bmatrix} T^{-1}$
    - $D^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_q^k \end{bmatrix}$
  - Transient response depends on the terms of the form λ_i^k

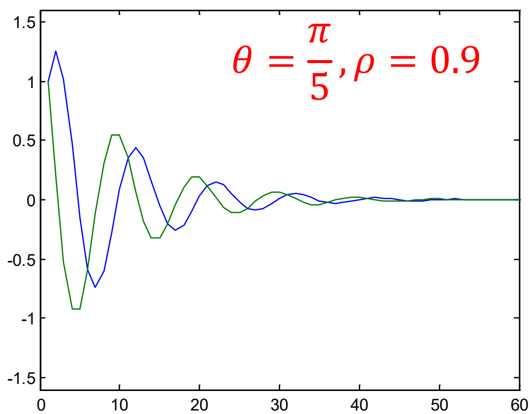
- **General case: Jordan form**
  - $A = TJT^{-1} = T\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix}T^{-1} \Rightarrow \; A^k =$
  - **Fact:** if $J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \Rightarrow J_i^k = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \frac{k(k-1)}{2}\lambda_i^{k-2} \\ 0 & \lambda_i^k & k\lambda_i^{k-1} \\ 0 & 0 & \lambda_i^k \end{bmatrix}$
- Transient response depends on the terms of the form

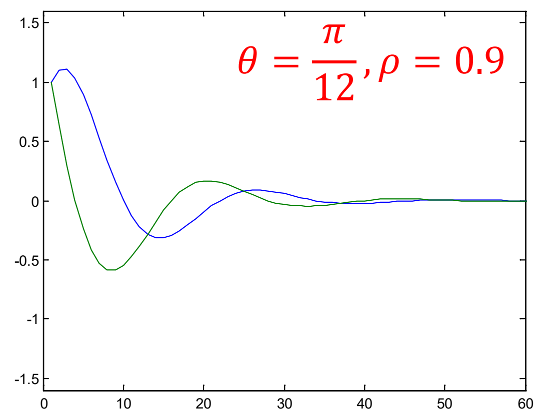
$$\frac{k(k-1)\cdots(k-j)}{j!}\lambda_i^{k-j}$$

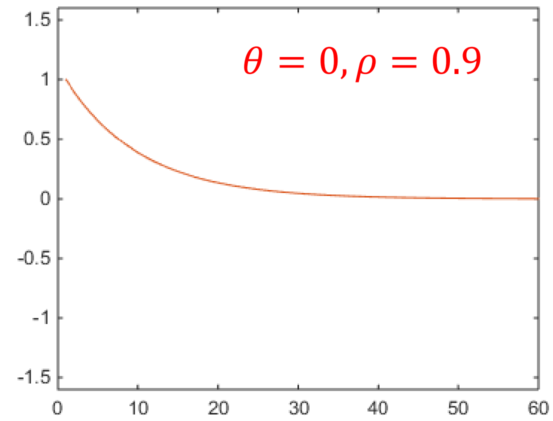
- The shape of transient response is determined by the locations of the eigenvalues

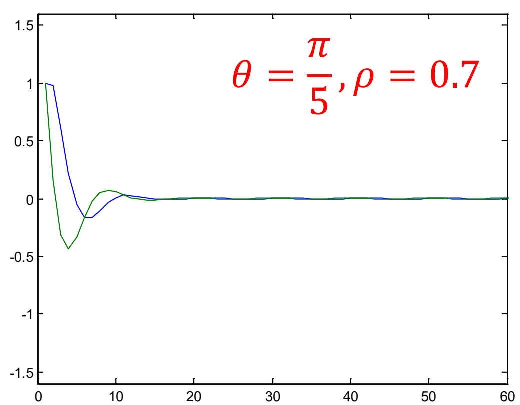
$$A = \rho \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \Rightarrow \lambda_{1,2} = \rho(\cos(\theta) \pm j\sin(\theta))$$

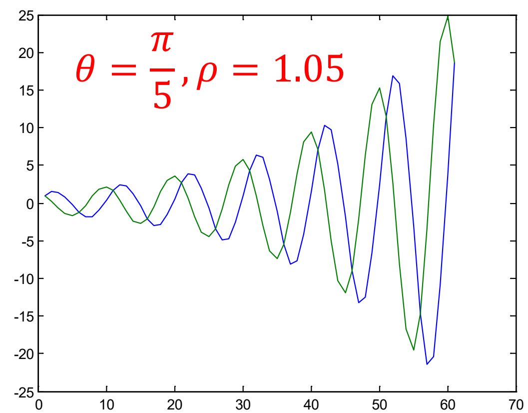
e.g: $x(k) = A^k x(0)$, with $x(0) = [1 \quad 1]^T$











- Large $|\lambda|$ produces slow convergence, while a small $|\lambda|$ produces fast convergence
- A real λ produces a monotonic response, while a complex λ produces an oscillatory response
- For a complex λ, the response becomes more oscillatory as the ratio $\left|\frac{Im(\lambda)}{Re(\lambda)}\right|$ increases
- Control design goal (for linear system): to modify the eigs of original system to achieve desired response.
- Feedback control fall into two categories
  - **State Feedback**: all state variables are measured and can be used in feedback
$$u(t) = g(x(t))$$
  - **Output Feedback**: Only output $y = Cx + Du$ (typically dim(y)<dim(x)) are measured and can be used in feedback
$$u(t) = g(y(t))$$

Outline

- Eigenvalues ↔ System Response
- **Full State-feedback: Eigenvalue Assignment**
- Luenberger Observer Design
- Output-feedback Control and Separation Principle

- **The concept of Closed-loop System**

- **The concept of closed-loop System**

- State feedback: full state information available to make control decision:
  - We focus on linear case: Let $u = -Kx$
  - we just need to design the feedback gain matrix K

- Plug in to obtain closed-loop system:
  - $x(k+1) = Ax(k) + Bu(k) =$
  - Closed-loop system matrix: (A-BK)
  - Pole placement (eigenvalue assignment) problem: find K so that the closed-loop system $A - BK$ has the desired set of eigenvalues

- **Single Input case:**
  - Consider **controllable canonical form**

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}, \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \bar{C} \ \ and\ \bar{D} \text{ arbitrary}$$

  - If a system $(\bar{A}, \bar{B})$ is in controllable canonical form, then it is always controllable (verify this by checking the controllability matrix of (A, B)

- Characteristic polynomial for $\bar{A}$

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}$$

  - $\Delta_{\bar{A}}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

- Characteristic polynomial for closed-loop $A_{cl} = \bar{A} - \bar{B}\bar{K}$
  - Assume: $\bar{K} = [k_1, k_2, \ldots, k_n]$

- $\Delta_{A_{cl}}(\lambda) = \lambda^n + (\alpha_{n-1} + k_n)\lambda^{n-1} + (\alpha_{n-2} + k_{n-1})\lambda^{n-2} + \cdots + (\alpha_1 + k_2)\lambda + (\alpha_0 + k_1)$

- Eigenvalue assignment: given desired $\lambda_1, \dots, \lambda_n$, how to choose $\bar{K}$?
  - **Step 1**: Find desired closed-loop characteristic polynomial:
    - $\Delta_{desired}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n + \alpha_{n-1}^* \lambda^{n-1} + \cdots + \alpha_1^* \lambda + \alpha_0^*$
  - **Step 2**: We know: $\Delta_{A_{cl}}(\lambda) = \lambda^n + (\alpha_{n-1} + k_n)\lambda^{n-1} + (\lambda_{n-2} + k_{n-1})\lambda^{n-2} + \cdots + (\alpha_1 + k_2)\lambda + (\alpha_0 + k_1)$

    choose $k_1, \dots, k_n$ to match coefficients

- Eigenvalue assignment example:

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \text{desired eig: } \lambda_1^* = 0.5, \ \lambda_2^* = -0.5$$

- What about general single input system (A, B), with $B \in R^{n\times 1}$
  - **Recall:** If original system: $x(k+1) = Ax(k) + Bu(k)$.
    Controllability matrix: $M_c = [B\ AB \cdots\ A^{n-1}B]$
  - Under similarity transformation: $x(k) = P\bar{x}(k)$, we have:
    $\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k)$, with $\bar{A} = P^{-1}AP, \bar{B} = P^{-1}B$
    $\bar{M}_c = \left[\bar{B}\ \bar{A}\bar{B}\ \cdots \bar{A}^{n-1}\bar{B}\right] = P^{-1}M_c$
  - FACT: $eig(A) = eig(\bar{A}),\ hence\ \Rightarrow \Delta_A(\lambda) = \Delta_{\bar{A}}(\lambda)$
- Main idea:
  - transform the system into a controllable canonical form $(\bar{A}, \bar{B})$
  - Design gain $\bar{K}$ to assign $eig(\bar{A} - \bar{B}\bar{K})$ to desired ones
  - Transform back to the original coordinate to get K so that eig(A-BK) = eig($\bar{A} - \bar{B}\bar{K}$)

- Eigenvalue assignment procedure for general single input system (A, B)
  - Step 1: Similarity transform: find P, such that $x(k) = P\bar{x}(k)$, and $\bar{x}(k)$ dynamic is in controllable canonical form

(1) Given A , compute: $\Delta_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

(2) We know: $\Delta_{\bar{A}}(\lambda) = \Delta_A(\lambda)$, by controllable canonical form structure, we have

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix}, \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

(3) Compute controllability matrix: $\bar{M}_c$ using $(\bar{A}, \bar{B})$ and M_c using (A, B)

$$\Rightarrow \quad P = M_c \bar{M}_c^{-1}$$

- Step 2: find $\bar{K}$ to assign desired eigs for $(\bar{A}, \bar{B})$

- Step 3: compute $K = \bar{K} P^{-1}$

  - Note that $A - BK$ and $\bar{A} - \bar{B}\bar{K}$ have the same set of eigs

- Coding Example: A =[2 0 -2; 4 -2 2; 0 2 -2], B = [1 0 1]';

- What about multiple inputs: ($B \in \mathrm{R}^{n\times m}, m \geq 2$)
  - Sometimes has redundancy, we can just use one column of B to assign eigs
  - General case is quite involved, use numerical tools to assign eigs or use LQR controller which will be covered later

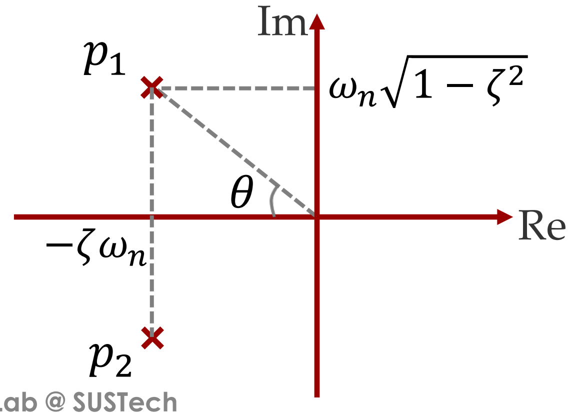
- Remarks on choosing desired poles (eigenvalues)
  - Continuous time case:

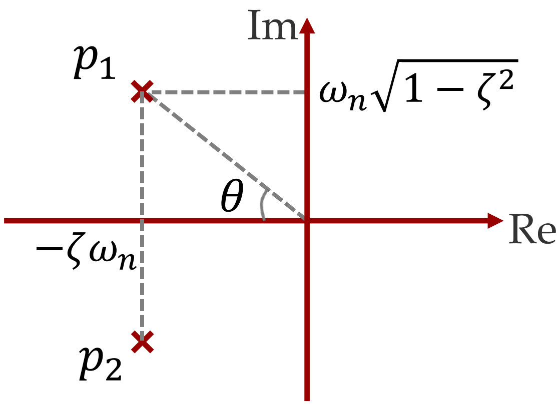
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

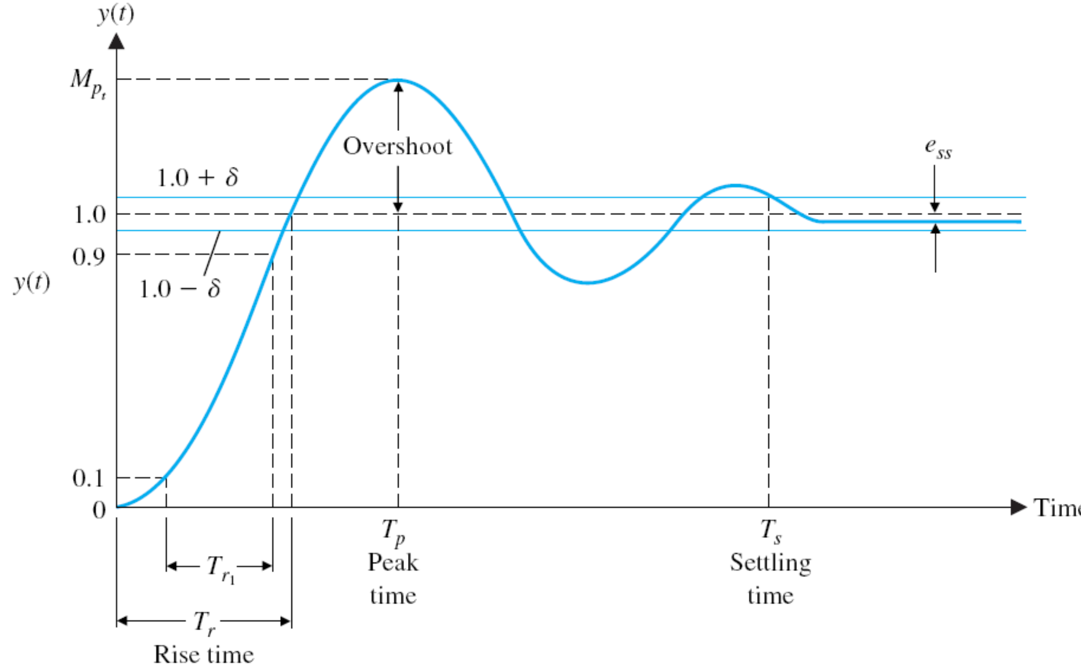
  - ω_n : natural frequency
  - ζ: damping ratio ($\zeta > 1$ overly-damped, $\zeta = 1$ critically damped, $\zeta < 1$ under-damped)

  - Under damped system ($\zeta < 1$): two complex poles:

$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, define: $\theta = \cos^{-1}\zeta$







$T_s = \frac{4}{\zeta\omega_n}$, **settling time**

$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$, **peak time**

PO$= 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$ **percent overshoot**

- Tradeoffs:

1. Poles moves to the left, i.e. larger $\zeta\omega_n$
2. Poles moves up, i.e., larger $\omega_n\sqrt{1-\zeta^2}$
3. Smaller θ

- Discrete time case:
  - Relations:
    - $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
    - T: sampling time
    - $z = e^{sT}$

- Pole selection example:
  - Suppose we want settling time $T_s \leq 5$ sec and $PO \leq 35\%$

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- **Observer Design:**
  - state vector is not available; u can only depend on output y
  - **Observer:** estimate system state vector $\hat{x}(k) \approx x(k)$ given $y(k), u(k)$ and (A, B, C, D)
  - **Key**: generate estimate iteratively according to known system dynamics:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \boldsymbol{L}\,[\boldsymbol{y(k)} - \boldsymbol{C\hat{x}(k)} - \boldsymbol{Du(k)}]$$

  - Iteratively update state estimate using previous estimate $\hat{x}(k)$ and new data available at time k: $u(k), y(k)$,
  - This way of estimating state is called Luenberger observer

- State estimation error vector: $e(k) = x(k) - \hat{x}(k)$
- Error dynamics: $e(k+1) = (A - LC)e(k)$

- **Goal**: design L matrix such that $eigs(A - LC)$ are at desired locations to ensure estimation error converge to zero with a desired transient

- **Observer design**: Find observer gain matrix $\boldsymbol{L}$ such that error dynamics have desired eigenvalues

- **Duality Theorem**: (A, C) observable $\Leftrightarrow$ $\left(A^T, C^T\right)$ controllable

  (remark: We say a pair (F, H) is controllable if a system with "A" matrix equal to F and "B" matrix equal to H is controllable. This also means $M_c = [H \quad FH \quad F^2H \quad \cdots \quad F^{n-1}H]$ is full rank)

- Consequence of the duality theorem: **If system (A,C) is observable**, we can use feedback gain design method to find observer gain L such that $eig(A-LC)$ has desired eigs

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- Output feedback control procedure:
  - System: $x(k+1) = A\,x(k) + Bu(k), y(k) = Cx(k) + Du(k)$
  - Find K, L such that $A - BK$ and $A - LC$ have desired eigs
  - At time $k = 0$, pick arbitrary $\hat{x}(0)$
  - For $k \geq 0$, given $\hat{x}(k), u(k), y(k)$, compute:
    - $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L\,[y(k) - C\hat{x}(k) - Du(k)]$
    - $u(k+1) = -K\hat{x}(k+1)$

- General guideline:

  eigenvalues of (A - BK) are chosen to meet the design specifications on the transient response. The eigenvalues of (A-LC) are chosen **much faster** than those of (A - BK)

- **Separation principle**: Observer eigs and controller eigs can be assigned separately
  - Closed-loop dynamics: joint state vector: $\begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$ with $u(k) = -K\hat{x}(k)$
  - Dynamics for joint state vector:

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} Ax(k) + Bu(k) \\ Ax(k) + Bu(k) - (A\hat{x}(k) + Bu(k) + L\,[y(k) - C\hat{x}(k) - Du(k)]) \end{bmatrix}$$

$$= \begin{bmatrix} Ax(k) + B(-K)\hat{x}(k) \\ (A - LC)e(k) \end{bmatrix} = \begin{bmatrix} Ax(k) - BKx(k) + BKe(k) \\ (A - LC)e(k) \end{bmatrix}$$

$$= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$$

  - The design of K and L can be done separately to meet specified controller and observer performance (characterized by eigs(A-BK) and eigs(A-LC))

- Summary