Fall 2022 ME424 Modern Control and Estimation

Lecture Note 9: Kalman Filter
- Extended Kalman Filter

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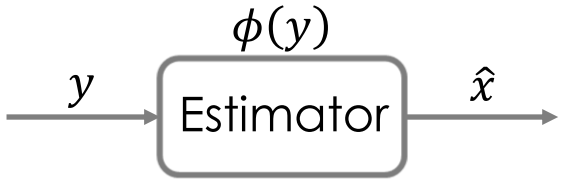
Recall:

- Suppose we want to estimate the value of a hidden random vector $X \in \mathrm{R}^n$ based on observations of a related vector $Y \in \mathrm{R}^m$.

- We have to know the relationship between X and Y. Suppose we take probabilistic viewpoint of their relations, namely, $(X, Y) \sim f_{XY}(x, y)$

- An estimator $\phi(y)$ is a function that maps each measurement $\mathrm{Y} = y$ to an estimate $\hat{x}$



- **MMSE Theorem**: The Minimum Mean-Squared Estimator for X given $Y = y$, that minimizes $E\left(||\phi(Y) - X||^2\right)$ is given by

$$\hat{X}_{MMSE} = \phi_{MMSE}(y) = E(X|Y = y)$$

Recall:

- Kalman filter is a recursive way to compute $E(x_k|Y_k)$ for linear Guassian system
- For nonlinear systems, we can use Extended Kalman Filter (EKF)
  - System setup: $$x_{k+1} = f(x_k, u_k) + w_k$$ $$y_k = h(x_k, u_k) + v_k$$
  - $x_k \in R^n$--- system state at time k
  - $y_k \in R^m$ --- measurement vector at time k
  - $Y_k \triangleq \begin{bmatrix} y_0^T & y_1^T & \dots & y_k^T \end{bmatrix}^T$ --- collection of measurements up to time k
  - $u_k \in \mathrm{R}^p$--- system input at time k (deterministic input)
  - $w_k \in R^n \sim N(0, Q_k)$, $v_k \in R^p \sim N(0, R_k)$
  - Assume $x_0 \sim N(\mu_0, \Phi_0)$, $x_0 \perp w_k, x_0 \perp v_k, w_k \perp v_j, \forall k, j, and$

$$w_k \perp w_j, \qquad v_k \perp v_j, \qquad \forall k \neq j$$

Preview of Extended Kalman Filter

- By fundamental theorem of estimation, we know that the MMSE is given by $E(x_k|Y_k)$
  - So we again needs to compute $E(x_k|Y_k)$
- With nonlinear dynamics, x_k is a random variable that **may not be Gaussian.**
- Extended Kalman Filter tries to
  - Approximate x_k as a Gaussian
  - Approximate the nonlinear dynamics as linear dynamics

- Notations: $\hat{x}_{k|k} = E(x_k|Y_k)$, $\quad \hat{x}_{k|k-1} = E(x_k|Y_{k-1})$

$$P_{k|k} = E\left(\left(x_k - \hat{x}_{k|k}\right)\left(x_k - \hat{x}_{k|k}\right)^T \middle| \mathrm{Y_k}\right)$$

$$P_{k|k-1} = E\left(\left(x_k - \hat{x}_{k|k-1}\right)\left(x_k - \hat{x}_{k|k-1}\right)^T \middle| Y_{k-1}\right)$$

Simplified notation: $\hat{x}_k \triangleq \hat{x}_{k|k}$, $\quad P_k = P_{k|k}$

- Goal: recursively compute:

$$\begin{Bmatrix} \hat{x}_k \\ P_k \end{Bmatrix} \longrightarrow \begin{Bmatrix} \hat{x}_{k+1|k} \\ P_{k+1|k} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \hat{x}_{k+1} \\ P_{k+1} \end{Bmatrix}$$

Extended Kalman Filter Derivation:

- **Step 1: Prediction (via linearization):**
  - Given $\hat{x}_k = E(x_k | Y_k), P_k = E\left((x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | Y_k\right),$
  - Need: $\hat{x}_{k+1|k} = E(x_{k+1} | Y_k),\ P_{k+1|k} = E\left(\left(x_{k+1} - \hat{x}_{k+1|k}\right)\left(x_{k+1} - \hat{x}_{k+1|k}\right)^T | Y_k\right)$
  - Recall the linear Gaussian case: $x_{k+1} = Ax_k + Bu_k + w_k,$

    the prediction step: $\hat{x}_{k+1|k} = A_k \hat{x}_k + B_k u_k, \quad P_{k+1|k} = A_k P_k A_k^T + Q_k$
  - EKF: Linearize $f(x, u)$ around the current state estimate $\hat{x}_k$ and input u_k

Summary of EKF Prediction Step:

Linearization: $F_k \triangleq \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_k, u_k}$

$$\hat{x}_{k+1|k} = f(\hat{x}_k, u_k), \quad P_{k+1|k} = F_k P_k F_k^T + Q_k$$

Extended Kalman Filter Derivation:

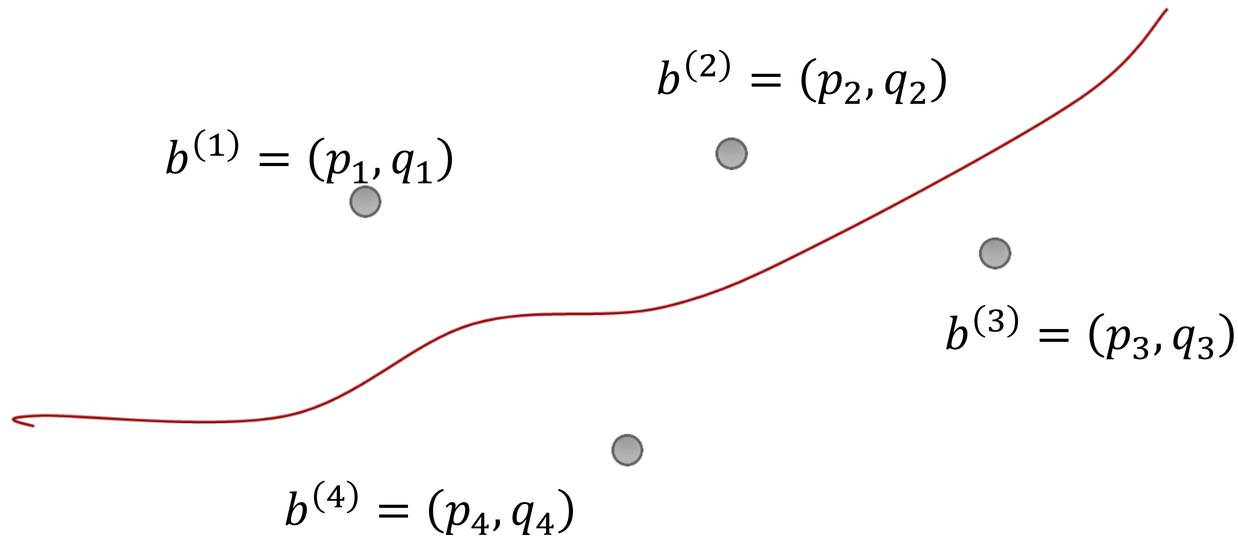
- Step 2: Measurement update through linearization:

Recall the linear case:

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y_k = Cx_k + Du_k + v_k$$

$$K_{k+1} = P_{k+1|k}C^T\left(CP_{k+1|k}C^T + R_{k+1}\right)^{-1}$$
$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - C\hat{x}_{k+1|k} - Du_{k+1}\right)$$
$$P_{k+1} = (I - K_{k+1}C)P_{k+1|k}$$

- ## Application Example I for EKF



- beacons with known positions $b^{(i)} = \left(b_1^{(i)}, b_2^{(i)}\right)$
- p_k: robot location at time k
- $y_{k,i}$: range measurement from beacon i at time k.
  - Typical measurement model: $y_{k,i} = \left|\left|b^{(i)} - p_k\right|\right| + v_i$
- Goal: find the best estimate of p_k given measurement $\{y_0, y_1, \dots, y_k\}$

- Derivation of the system model under constant speed assumption

Here, we want to use dynamics information in addition to the beacon measurement. we assume constant speed motion model:

- **EKF derivation and implementation**

Application Example II : Joint State and Parameter Estimation

- Consider a 2nd-order continuous time system:

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = \omega_n^2\, u(t)$$

- System state: $x = [y, \dot{y}]^T$, system parameter $\theta = [\xi, \omega_n]^2$
- System input-output u, y
- Question: How to use (u, y) data to jointly estimate x and θ?