ME424 Fall 2022 | **Homework # 6**
Assigned December 11, 2022 | Due December 17, 2022

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1. Let $X = [X_1\ X_2\ X_3]^T$ be a zero mean 3D Gaussian random vector with covariance

$$\Sigma_X = \begin{bmatrix} 2 & 0 & \sigma_{13} \\ 0 & 2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 2 \end{bmatrix}$$

   (a) Suppose $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 1$. Let $W = [X_1\ X_2]^T$ and $Z = X_3$. Use the conditional mean formula for Gaussian random vectors discussed in class to compute the conditional mean and covariance of W given $Z = 10$. Without calculation, can you guess whether the conditional mean of W (each entry) is positive or negative if you observe $Z = -10$ instead. Briefly explain your answer.

   (b) Is X_1 independent of X_2? Under the same numerical values given in part (i), is $X_1|Z = 10$ independent of $X_2|Z = 10$? Dose your answer change if $\sigma_{13} = \sigma_{31} = 0$ and $\sigma_{23} = \sigma_{32} = 1$

   (c) Write down the probability density function for $W|Z = 10$. Use MATLAB "surf" function to plot the density function. Attach your code and plot.

2. Let $X = [X_1\ X_2\ X_3]^T$ be a 3D discrete random vector taking values in

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

   with probabilities $0.1, 0.4, 0.05, 0.25, 0.2$, respectively.

   (a) Find the conditional distribution (probability mass function): $p(x_1|X_2 = -1, X_3 = 3)$ for the conditional variable $X_1|(X_2 = -1, X_3 = 3)$ using the following two approaches:

      i. (Direct Approach): Find $Prob(X_2 = -1, X_3 = 3)$ and directly compute the conditional probabilities $p(x_1|X_2 = -1, X_3 = 3)$ for all possible x_1 values.

      ii. (Indirect Approach): (i) First compute probability mass function for the joint $(X_1, X_3)|X_2 = -1$, i.e., $p(x_1, x_3|X_2 = -1)$. For this you need to identify the possible pairs of values (X_1, X_3) may take and their probabilities under the condition that $X_2 = -1$. (ii) Then compute the marginal probability $Prob(X_3 = 3|X_2 = -1)$ using the joint distribution computed in part (i); (iii) Then using the results in the previous two parts to compute the $p(x_1|X_2 = -1, X_3 = 3)$

(b) Compute the MMSE for X_1 given the observations that $X_2 = -1, X_3 = 3$.

3. $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\mu, \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right)$, where μ is random with $Prob\left(\mu = \mu^{(1)}\right) = 0.2$ and $Prob\left(\mu = \mu^{(2)}\right) = 0.8$. Here, $\mu^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mu^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the MSSE of X given $Y = 1$

4. Let X, V be independent Gaussians with $X \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right)$ and $V \sim N(0, 3)$. Given a measurement model $Z = HX + V$ with $H = \begin{bmatrix} 1 & 2 \end{bmatrix}$. Compute $E(X|Z = 4)$ and $Cov(X|Z = 4)$ using the following two approaches

   (a) Using conditional Gaussian formula

   (b) Using Kalman filter formula