1. **Air Fair**: Air transportation is available between all pairs of n cities, but because of a perverse fare structure, it may be more economical to go from one city to another through immediate stops. A cost-minded traveler wants to find the minimum cost fare to go from an origin city s to a destination city t. The airfare between cities i and j is denoted by a_{ij}, and for every intermediate stop, there is a stopover cost c (a_{ij} and c are assumed positive). Thus, for example, to go from s to t directly it costs a_{st}, while to go from s to t with intermediate stops at cities i_1 and i_2, it costs $a_{si_1} + c + a_{i_1 i_2} + c + a_{i_2 t}$.

   (a) Formulate the problem of finding the cheapest path from s to t as a dynamic programming problem. Make sure to specify the number of total stages (horizon length), terminal cost, running cost, control constraint set, and state update equation (note that there are multiple ways to write the state update equation). Please also write down the value iteration formula and the optimal control law in this case.

   (b) Write a Python function that takes a_{ij}, c, s, and t as input arguments, and returns the cheapest path and the corresponding cost. Test your program using the data given the attached file. The matrix A is the attached file specifies the pair-wise cost a_{ij} for traveling from city i to city j among a 10-city network. Assume that $c = 1$. Find the optimal trip and the corresponding optimal cost from city 8 to city 9.

2. **LQR Controller:** Given a discrete time state space model with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

   Suppose your design objective is represented by a state weighting matrix and control weighting matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{and } R = 2 \tag{2}$$

   (a) Compute value iterations (P_k, $k = 1, 2, \ldots$) using the Riccati recursion formula starting with $P_0 = Q$. Find the converged matrix P^* and the optimal control gain matrix K^*. (you cannot directly use "dlqr" function).

   (b) For initial state $x_0 = [1, -1, 1]^T$, compare the performance of the optimal gain matrix K^* with respect to two stabilizing feedback gains designed using the eigenvalue assignment approach (you can pick desired eigenvalue sets of your choice, and you are allowed to use "place" function). You need to plot trajectories of x_1 and x_2 (no need for x_3) for the comparison. You also need to compare the actual cost over 1000 steps (hint: simulate the closed-loop system under different feedback gains for 1000 steps, and compare their costs $J_{1000}(x_0) = \sum_{k=0}^{999} \left(x_k^T Q x_k + u_k^T R u_k\right)\big)$.

(c) Compute the 10-horizon optimal cost for initial state $x_0 = [1, 0, 1]^T$ (namely, $V_{10}(x_0)$). Find the 10-horizon optimal control sequence u_k^*, $k = 0, 1, \ldots, 10$.