MEE5114 Advanced Control for Robotics

Lecture 1: Linear Differential Equations and Matrix Exponential

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Outline

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

Motivations

- Most engineering systems (including most robotic systems) are modeled by Ordinary Differential (or Difference) Equations (ODEs)

- Example: Dynamics of 2R robot

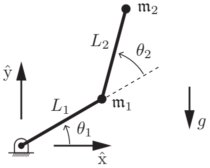
$$\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta,\dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})},$$

with

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix},$$



- Screw theory, exponential coordinate, and Product of Exponential (PoE) are based on the (linear) differential equation view of robot kinematics

Linear Differential Equations (Autonomous)

- Linear Differential Equations: ODEs that are linear wrt variables e.g.:

$$\begin{cases} \dot{x}_1(t) + x_2(t) = 0 \\ \dot{x}_2(t) + x_1(t) + x_2(t) = 0 \end{cases} \qquad \begin{cases} \ddot{y}(t) + z(t) = 0 \\ \dot{z}(t) + y(t) = 0 \end{cases}$$

- State-space form (1st-order ODE with vector variables):

General Linear Control Systems

- General (Autonomous) Dynamical Systems: $\dot{x}(t) = f(x(t))$
  - $x(t) \in \mathbb{R}^n$: state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$: *vector field*

- Non-autonomous: $\dot{x}(t) = f(x(t), t)$

- Control Systems: $\dot{x}(t) = f(x(t), u(t))$
  - vector field $f : \mathbb{R}^n \times \mathbb{R}^m$ depends on external variable $u(t) \in \mathbb{R}^m$

- General Linear Control Systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

  - $x \in \mathbb{R}^n$: system state, $u \in \mathbb{R}^m$: control input, $y \in \mathbb{R}^p$: system output
  - A, B, C, D are constant matrices with appropriate dimensions

Existence and Uniqueness of ODE Solutions

- Function $g : \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* over domain $\mathcal{D} \subseteq \mathbb{R}^n$ if $\exists\, L < \infty$

$$\|g(x) - g(x')\| \leq L\|x - x'\|, \forall x, x' \in \mathcal{D}$$

- **Theorem [Existence & Uniqueness]** Nonlinear ODE

$$\dot{x}(t) = f(x(t), t), \quad \text{I.C. } x(t_0) = x_0$$

has a *unique* solution if $f(x, t)$ is Lipschitz in x and piecewise continuous in t

Existence and Uniqueness of Linear Systems

- **Corollary**: Linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

  has a unique solution for any piecewise continuous input $u(t)$

- *Homework*: Suppose A becomes time-varying $A(t)$, can you derive conditions to ensure existence and uniqueness of $\dot{x}(t) = A(t)x(t) + Bu(t)$?

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How to Solve Linear Differential Equations?

- General linear ODE: $\dot{x}(t) = Ax(t) + d(t)$
- The key is to derive solutions to the autonomous linear case: $\dot{x}(t) = Ax(t)$, with $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and initial condition (IC) $x(0) = x_0$.
- By existence and uniqueness theorem, the ODE $\dot{x} = Ax$ admits a unique solution.
- It turns out that the solution can be found analytically via the *Matrix Exponential*

What is the "Euler's Number" e?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution:

- What is the number "e"?

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- This can be extended to complex variables:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

  This power series is well defined for all $z \in \mathbb{C}$

- In particular, we have $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

- This power series is well defined for any finite square matrix $A \in \mathbb{R}^{n \times n}$.

Some Important Properties of Matrix Exponential

- $Ae^A = e^A A$
- $e^A e^B = e^{A+B}$ if $AB = BA$
- If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$
- For every $t, \tau \in \mathbb{R}$, $e^{At} e^{A\tau} = e^{A(t+\tau)}$
- $\left(e^A\right)^{-1} = e^{-A}$

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Autonomous Linear Systems

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.
- With the definition of matrix exponential, we can show that the solution to (2) is given by
$$x(t) = e^{At} x_0$$

Computation of Matrix Exponential (1/2)

- Directly from definition

- For diagonalizable matrix:

Computation of Matrix Exponential (2/2)

- Using Laplace transform

Solution to General Linear Systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3}$$

- $x \in \mathbb{R}^n$ is system state, $u \in \mathbb{R}^m$ is control input, $y \in \mathbb{R}^p$ is the system output
- A, B, C, D are constant matrices with appropriate dimensions
- **Homework:** The solution to the linear system (3) is given by

$$\begin{cases} x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{cases}$$

More Discussions