ME5114 Sp23 **Homework # 1**

Assigned Feb 16, 2023 Due: Feb 23 , 2023

- This homework is mainly about the required math and Python coding background of this course. Depending on your previous training, you may need to read related materials online. Some background materials and tutorials are uploaded to Blackboard and course website
- You can type your solution in Latex (encouraged but not required) or just hand write the solution on paper. Submit your homework (pdf) through Blackboard
- To receive credits, please write down all the necessary steps leading to the final answer.

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1. **Python**: Please go over the Python tutorial posted on the course website. Make sure you can write basic python codes.

2. **Lipschitz Continuity**

   (a) Please state the formal definition of continuous functions

   (b) Please state the formal definitions of Lipschitz continuity and locally Lipschitz continuity.

3. **Matrix calculus**

   (a) Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be a scalar function of matrix variable. Please write a tutorial paragraph explaining (in your own words) the meaning of $\frac{\partial}{\partial X} f(X)$.

   (b) Let $A \in \mathbb{R}^{n\times m}, X \in \mathbb{R}^{m\times n}$. Derive an expression for $\frac{\partial}{\partial X} tr(AX)$ (show your derivation steps; your derivation should be directly from the definition of matrix derivatives)

   (c) Derive an expression for $\frac{\partial}{\partial x} f(x)$, where $f(x) = x^T Q x + tr(xx^T)$ and $x \in \mathbb{R}^n$

4. **Inner product**

   (a) Describe the way to calculate the angle between two vectors $x, y \in \mathbb{R}^n$ using inner product

   (b) Trace can be used to define inner products for matrices. Let $A, B \in \mathbb{R}^{m\times n}$, then $<A, B>= tr(A^T B)$. Compute the angle between the following two matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

5. **Some linear algebra**

   (a) State the condition on A such that $Ax = b$ has at least one solution

   (b) Let $A = [a_1, a_2, a_3, a_4]$, where $a_i \in \mathbb{R}^n$ are columns of A. Suppose a_1, a_2 are linearly independent, and $a_3 + a_1 = a_2$ and $a_4 - a_3 = a_1$. Compute $rank(A)$ and $Null(A)$

   (c) Given a vector $y \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times m}$, find an expression of the projection of y onto the column space of A

6. **Ellipsoids:** Ellipsoid in $\mathbb{R}^n$ have two equivalent representations: (i) $E_1(P, x_c) = \{x \in \mathbb{R}^n : (x - x_c)P^{-1}(x - x_c) \le 1\}$ and (ii) $E_2(A, x_c) = \{Au + x_c : \|u\|^2 \le 1\}$. Given an eillipsoid $E_1(P, x_c)$ with P positive definite, its volume is $\nu_n \sqrt{det(P)}$ where ν_n is the volume of unit ball in $\mathbb{R}^n$, its semi-axes directions are given by the eigenvectors of P and the lengths of semi-axes are $\sqrt{\lambda_i}$, where λ_i are eigenvalues of P.

(a) Given an Ellipsoid $E_1(P, x_c)$, find the corresponding (A, b) (in terms of P and x_c) such that $E_2(A, b)$ represents the same ellipsoid as $E_1(P, x_c)$

(b) Draw the ellipse $E_1(P, x_c)$ with $P = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and $x_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by hand.

7. **Linear System Solution**: Consider the following linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{with } x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input. Show

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

is a solution to the above control system.