MEE5114 Advanced Control for Robotics

Lecture 2: Rigid Body Configuration and Velocity

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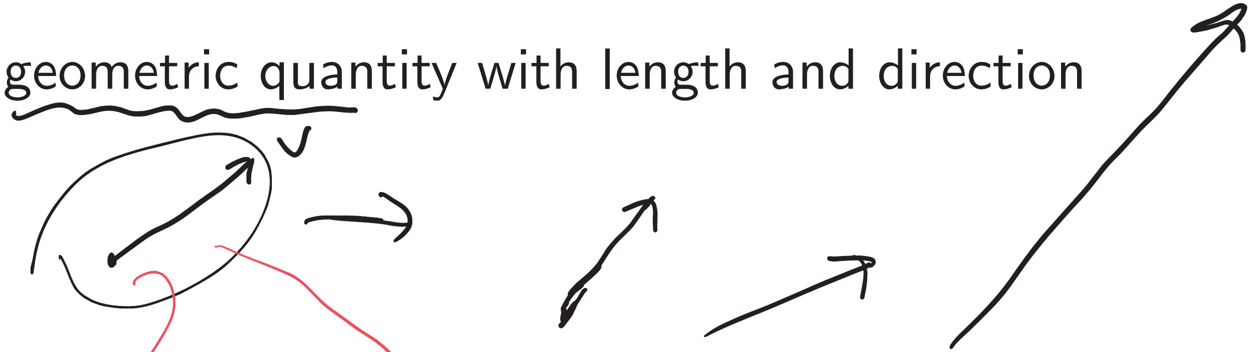
Outline

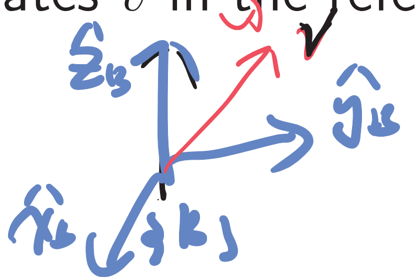
- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

Free Vector

- **Free Vector**: geometric quantity with length and direction



- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented by its coordinates v in the reference frame.



- v denotes the physical quantity while ${}^{A}v$ denote its coordinate wrt frame $\{A\}$.

Frame: coordinate sys based on basis vectors

$\{A\}$-frame: $\{\hat{x}_A, \hat{y}_A, \hat{z}_A\}$, $\{o_A\}$

${}^{A}v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ means

$v = 1 \cdot \hat{x}_A + 2 \cdot \hat{y}_A + 3\hat{z}_A$

${}^{A}\hat{x}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ${}^{B}\hat{x}_A$

Point

$V_3 = V_1 + V_2$ ← "coordinate-free"

But we can express "physics" in different coordinate frames.

choose A: ${}^{A}V_3 = {}^{A}V_1 + {}^{A}V_2$

choose B: ${}^{B}V_3 = {}^{B}V_1 + {}^{B}V_2$

$\boxed{{}^{A}V_3 \neq {}^{B}V_1 + {}^{B}V_2}$

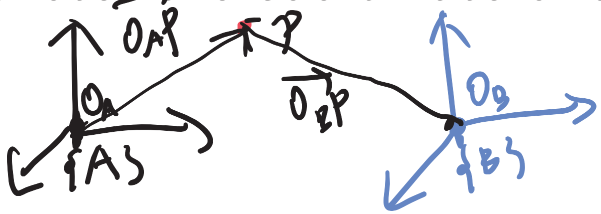
- **Point**: p denotes a point in the physical space

  $\bullet\ p$

- A point p can be represented by a vector from frame origin to p

- ${}^{A}p$ denotes the coordinate of a point p wrt frame $\{A\}$

${}^{A}p = {}^{A}(\overrightarrow{O_A p})$



${}^{B}p = {}^{B}(\overrightarrow{O_B p})$

$\overrightarrow{O_B p} \neq \overrightarrow{O_A p}$

- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector for which the reference frame is clear from the context.

${}^{A}(\overrightarrow{O_B p})$

Cross Product

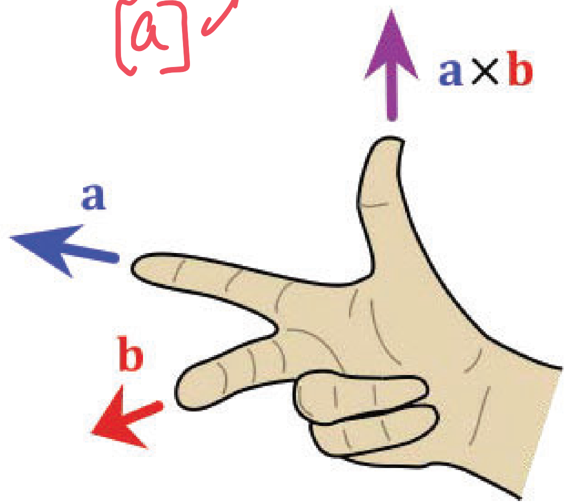
- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$\overset{\in\mathbb{R}^3}{a} \times \overset{\in\mathbb{R}^3}{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{[a]^{\wedge}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Properties:

- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$



Skew symmetric representation

- It can be directly verified from definition that $a \times b = [a]b$, where

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

symmetric: $A = A^T$

skew symmetric: $A = -A^T$

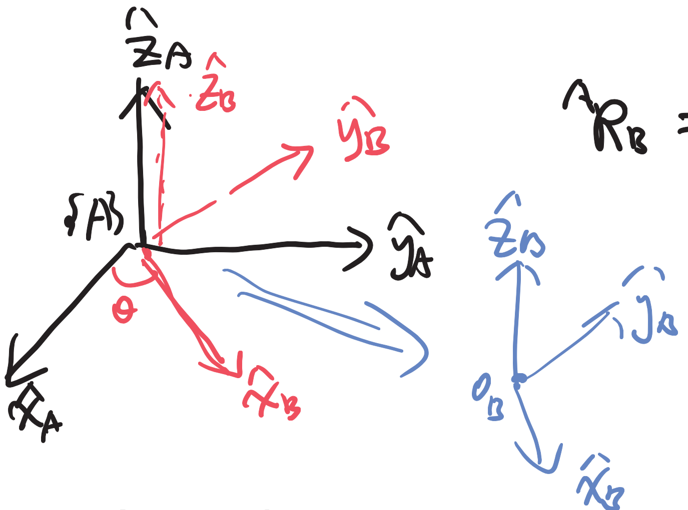
- $[a] = -[a]^T$ (called skew symmetric)
- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin O
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}} \Leftarrow$ right hand rule

- **Rotation Matrix**: specifies orientation of one frame relative to another

$$ {}^A R_B \triangleq \left[\ {}^A\hat{x}_B \quad {}^A\hat{y}_B \quad {}^A\hat{z}_B\ \right] $$



$$ {}^A R_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

- A valid rotation matrix R satisfies: (i) $R^T R = I$; (ii) $\det(R) = 1$

$$ \begin{bmatrix} {}^A\hat{x}_B^T \\ {}^A\hat{y}_B^T \\ {}^A\hat{z}_B^T \end{bmatrix} \left[\ \ \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ \det({}^A R_B) = {}^A\hat{x}_B^T \left({}^A\hat{y}_B^T \times {}^A\hat{z}_B \right) = 1 $$

Special Orthogonal Group

$R \in SO(3)$

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

prove

$$\leftrightarrow \quad SO(n) = \{R \in \mathbb{R}^{n\times n} : \boxed{R^T R = I, \det(R) = 1}\}$$

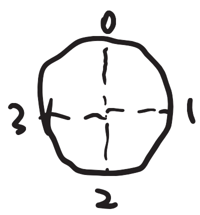
- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:

$(\mathbb{Z}, +)$, clock arithmetic.

axioms

- **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
- **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a, b, c \in G$
- **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
- **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.



$\{\{0, 1, 2, 3\}, \text{"+"}\}$

$0+2=2$

$2+2=0$

$2+3=1$

"0"

$\mathrm{inv}(2)=2$

$\mathrm{Inv}(3)=1$

geometry: study of "symmetry"

Use of Rotation Matrix (1/2)

directly from definition

- Representing an orientation ${}^A R_B$: orientation of $\{B\}$ relative to A

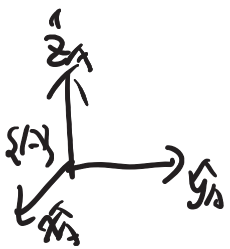
- Changing the reference frame ${}^A R_B$: Given vector v, its coordinates in $\{A\}$, $\{B\}$ are ${}^A v$, ${}^B v$

· ${}^A v = {}^A R_B \, {}^B v$: proof: "Coordinate-free"

· we have only one vector: v, suppose ${}^A v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, ${}^B v = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$

"physics": $v = \underline{\alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A}$

$v = \underline{\beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B}$



$\Rightarrow \quad \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B \quad \ldots$ "physics"

$\Rightarrow$ state/express physics in $\{A\}$:

$\Rightarrow \quad \alpha_1 {}^A\hat{x}_A + \alpha_2 {}^A\hat{y}_A + \alpha_3 {}^A\hat{z}_A = \beta_1 {}^A\hat{x}_B + \beta_2 {}^A\hat{y}_B + \beta_3 {}^A\hat{z}_B$

Use of Rotation Matrix (2/2)

$$\underbrace{\begin{bmatrix} {}^A\hat{x}_A & {}^A\hat{y}_A & {}^A\hat{z}_A \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{{}^A v} = \underbrace{\begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix}}_{{}^A R_B} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{{}^B v}$$

$${}^A v = {}^A R_B \, {}^B v$$

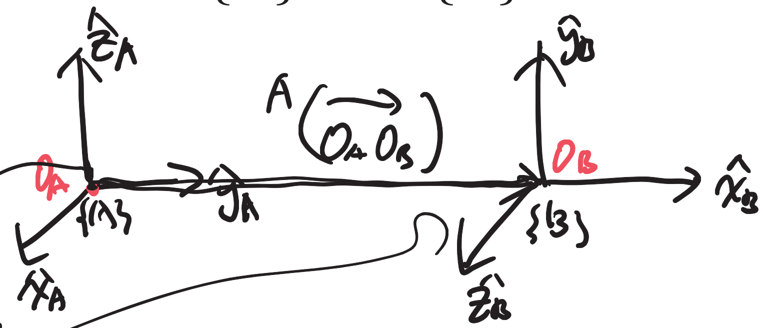
- Rotating a vector or a frame $\mathrm{Rot}(\hat{\omega}, \theta)$: will be discussed in next lecture.

"action" operator view of rotation matrix.

verb.

Rigid Body Configuration

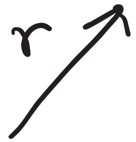
- Given two coordinate frames {A} and {B}, the configuration (pose) of B relative to A is determined by
  - ${}^A R_B$ and ${}^A o_B$



$${}^A R_B = [{}^A\hat x_B \;\; {}^A\hat y_B \;\; {}^A\hat z_B]$$

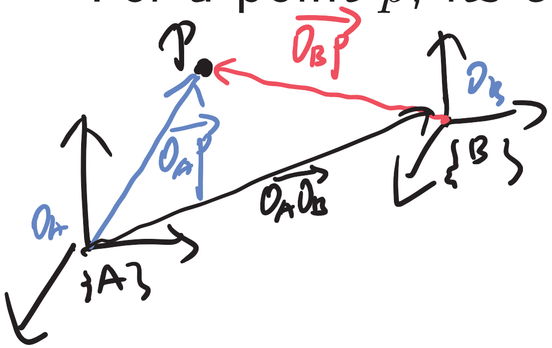
$${}^A o_B = {}^A(\overrightarrow{O_A O_B})$$

- For a (free) vector "r", its coordinates ${}^A r$ and ${}^B r$ are related by:



$${}^A r = {}^A R_B \, {}^B r \quad ✓$$

- For a point p, its coordinates ${}^A p$ and ${}^B p$ are related by:



"coordinate free":

$$\overrightarrow{O_A p} = \overrightarrow{O_A O_B} + \overrightarrow{O_B p} \Longrightarrow$$

choose {A} frame to express "physics"

$${}^B(\overrightarrow{O_B p}) = {}^B p$$

$$\underbrace{{}^A(\overrightarrow{O_A p})}_{{}^A p} = \underbrace{{}^A(\overrightarrow{O_A O_B})}_{{}^A o_B} + \underbrace{{}^A(\overrightarrow{O_B p})}_{{}^A R_B\, {}^B p}$$

$${}^A p = {}^A o_B + {}^A R_B \, {}^B p \quad \cdots (1)$$

Homogeneous Transformation Matrix

- **Homogeneous Transformation Matrix**: ${}^{A}T_{B}$

~~By~~ :

$$\underbrace{\underbrace{{}^{A}p}_{3\times 1} = \underbrace{{}^{A}O_B}_{3\times 1}}\ + \underbrace{{}^{A}R_B}_{3\times 3}\ \underbrace{{}^{B}p}_{3\times 1} \Rightarrow \boxed{\begin{bmatrix} {}^{A}p \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^{A}R_B & {}^{A}O_B \\ \underset{1\times 3}{0} & 1 \end{bmatrix}}_{4\times 4} \begin{bmatrix} {}^{B}p \\ 1 \end{bmatrix}}$$

3×1

$${}^{A}T_B \triangleq \begin{bmatrix} {}^{A}R_B & {}^{A}O_B \\ 0 & 1 \end{bmatrix}, \quad T = (R, p).$$

pose of a frame $\{B\}$ relative to $\{A\}$

- Homogeneous coordinates:

Given a point $p \in \mathbb{R}^3$, its homogeneous coordinate is defined as

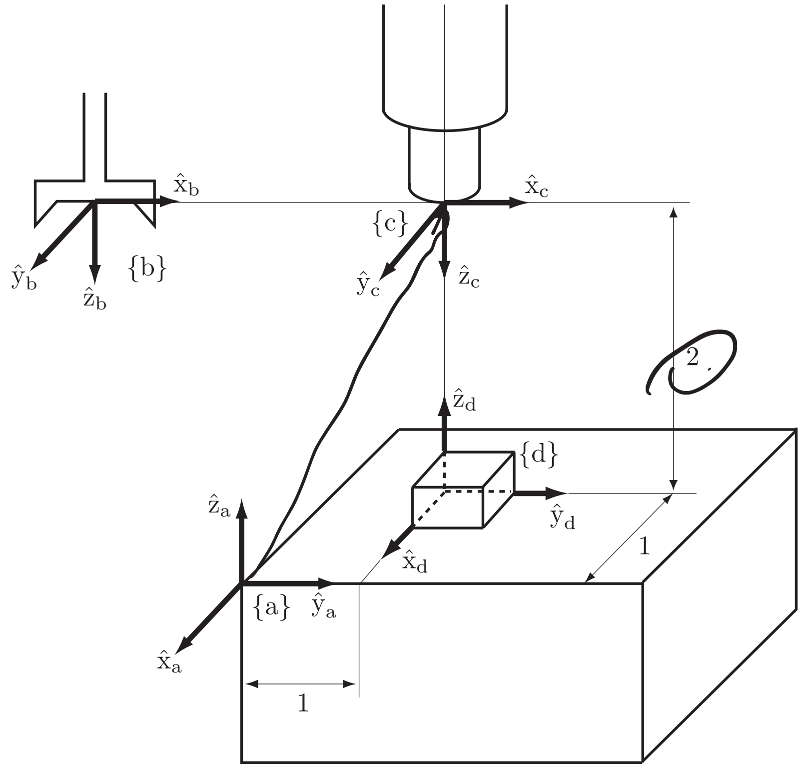
$$\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix} \in \mathbb{R}^4 \Rightarrow {}^{A}\tilde{p} = {}^{A}T_B\, {}^{B}\tilde{p}$$

Given a vector $v \in \mathbb{R}^3$, its homo-coord is $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$$v = p_1 - p_2 \qquad \tilde{v} = \tilde{p}_1 - \tilde{p}_2 = \begin{bmatrix} v \\ 0 \end{bmatrix}, \quad {}^{A}\tilde{v} = {}^{A}T_B\, {}^{B}\tilde{v}$$

Example of Homogeneous Transformation Matrix

Fixed frame $\{a\}$; end effector frame $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$. Suppose $\|p_c - p_b\| = 4$



1: Camera "location"? ${}^aT_c = ({}^aR_c, {}^ap_c)$

$${}^aR_c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad {}^ap_c = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$${}^aT_c = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2: end-effector frame ${}^aT_b$

$${}^cT_b = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

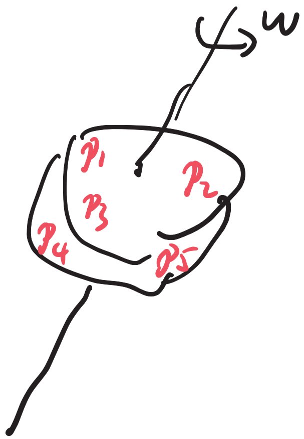
$${}^aT_b = {}^aT_c \, {}^cT_b$$

Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist) spatial vector
- Geometric Aspect of Twist: Screw Motion

Rigid Body Velocity (1/3)

- Consider a rigid body in motion. The body has infinitely many points $\{p_i\}$ with different velocities $\{v_{p_i}\}$



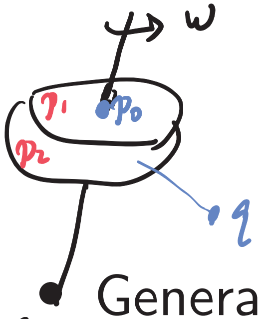
$$v_{p_1} = g(p_1, \text{para})$$
$$v_{p_2} = g(p_2, \text{para})$$
$$v_{p_3} = g(p_3, \text{para})$$
$$\vdots \qquad \vdots$$

parameter common to all points on the body.

  - All these velocities v_{p_i}'s are not independent
  - They can be expressed by the same set of parameter
  - Rigid body velocity (i.e. spatial velocity, twist) is one such parameterization

Rigid Body Velocity (2/3)

- Pure rotation case :



Assume p_0 on the rotation axis.

$$v_{p_i} = \boxed{\underbrace{\omega}_{\text{angular velocity}} \times \overrightarrow{p_0 p_i}} = g(p_i, \omega)\,, \qquad v_{p_0} = 0$$

- General motion

1°: Again, assume p_0 on rotation axis / body-fixed.

$$v_{p_i} = \dot{p}_i(t) = \left(p_0(t) + \overrightarrow{p_0 p_i}(t)\right)' = \underbrace{\boxed{v_{p_0}} + \omega \times \overrightarrow{p_0 p_i}}_{①} = g(p_i, \text{para})$$

para

In this case, p_0 is a reference point we use to express velocities of all other points.

2°. What if ref point NOT on rotation axis?

e.g. consider arbitrary body-fixed q, with velocity v_q

- we still have the same expression $\boxed{v_{p_i} = v_q + \omega \times \overrightarrow{q p_i}}$ ②

Why? use p_0 as intermediate variable.

q is body-fixed, $\Rightarrow$ by ① $\Rightarrow v_q = v_{p_0} + \omega \times \overrightarrow{p_0 q} = v_{p_i} - \omega \times \overrightarrow{p_0 p_i} + \omega \times \overrightarrow{p_0 q}$

$$\Rightarrow v_{p_i} = v_q + \omega \times \overrightarrow{q p_i}$$

Rigid Body Velocity (3/3)

- 3. What if the ref point "r" is NOT body-fixed. (e.g. r is stationary in space, or move in other way).

  – let q be body-fixed point "$\underbrace{\text{currently}}_{\text{time } t}$" "coincides" with r

i.e. $q(t) = r$ at time t ($q(t_1)$ may not equals r at $t_1 \neq t$)

By ②: $v_{p_i} = v_{q(t)} + \omega \times \overrightarrow{q(t)p_i(t)}$

If we define: "v_r" $\triangleq v_{q(t)}$, then $v_{p_i} =$ "v_r" $+ \omega \times \overrightarrow{rp_i}$ (same as ②)

"v_r": in representing rigid body B velocity
means the velocity of body-fixed point current coincides with r.

rigid body velocity:

Rigid Body Velocity: Spatial Velocity (Twist)

- How to represent a rigid body velocity?
  - Pick an arbitrary point r (reference point), which may or may not be body-fixed
  - Define v_r as the velocity of **the body-fixed point currently coincides with** r
  - For any body-fixed point p on the body: $\boxed{v_p = \text{``}v_r\text{''} + \omega \times (\overrightarrow{rp})}$ ← coordinate free.

of "body"

- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\vec{\omega}, \vec{v}_r) \in \mathbb{R}^6$

- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point r

- A rigid body $\mathcal{B}$ with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at v_r while rotating with angular velocity ω about an axis passing through r
  - This is just one way to interpret the motion.

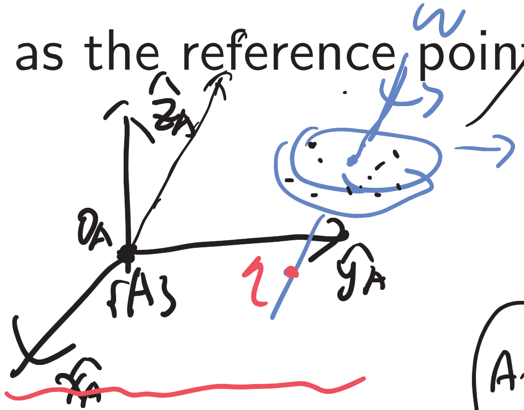
Spatial Velocity Representation in a Reference Frame

- Given frame $\{A\}$ and a spatial velocity $\mathcal{V}$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v_{o_A} \end{bmatrix}$$

convention

- Choose o_A (the origin of $\{A\}$) as the reference point to represent the rigid body velocity



- Coordinates of $\mathcal{V}$ in $\{A\}$: $^A\mathcal{V}_{o_A} = (^A\omega, {}^Av_{o_A})$

$$^A\mathcal{V}_{o_A} = \begin{bmatrix} ^A\omega \\ ^Av_{o_A} \end{bmatrix}$$

- By default, we assume the origin of the frame is used as the reference point: $^A\mathcal{V} = {}^A\mathcal{V}_{o_A}$

(ω, v_{o_A})

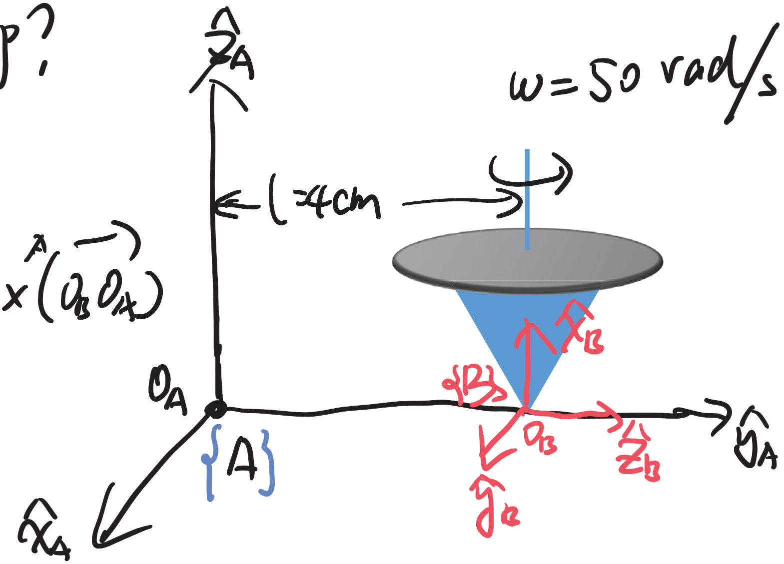
$^A\mathcal{V}$

let p be a body-fixed point of body.

$$v_p = v_{o_A} + \omega \times \overrightarrow{o_A p} \xRightarrow{\text{using } \{A\}} {}^Av_p = {}^Av_{o_A} + {}^A\omega \times {}^Ap$$

Example of Twist I

- Example I: What's the twist of the spinning top?



choose {A}-frame:

$$^A\mathcal{V}_{top} = \begin{bmatrix} ^A\omega \\ \hline ^Av_{O_A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \text{ rad/s} \\ \hline \\ \end{bmatrix}$$

$$^Av_{O_A} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$^Av_{O_A} = \underbrace{^Av_{O_B}}_{0} + {}^A\omega \times ({}^A\overrightarrow{O_B O_A})$$

$$= \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} \times \begin{bmatrix} 0 \\ -0.04 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

choose {B}-frame

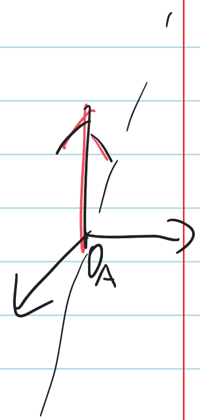
$$^B\mathcal{V}_{top} = \begin{bmatrix} ^B\omega \\ ^Bv_{O_B} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$$

Revisit: using ^AX_B : $\Rightarrow$ $^A\mathcal{V}_{top} = {}^AX_B\, {}^B\mathcal{V}_{top}$

for this example: $^AX_B = [Ad_T] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline & * & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \end{array}\right]$

$$* = \begin{bmatrix} 0 & 0 & 0.04 \\ 0 & 0 & 0 \\ -0.04 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$T = (R, p)$, $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 0 \\ 0.04 \\ 0 \end{bmatrix}$



$$V_p = V_q + \omega \times (\vec{qp}) = \omega \times (\vec{qp})$$

$$V_p = V_{O_A} + \omega \times (\vec{O_A p})$$

$$(V_{O_A}, \omega)$$

Example of Twist II

- Example II: what's ${}^b\mathcal{V}_{car}$ ${}^s\mathcal{V}_{car}$

car rotate about r with angular velocity w

$${}^b\mathcal{V}_{car} = \begin{bmatrix} {}^b\omega \\ {}^b v_{O_b} \end{bmatrix}, \quad {}^b\omega = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad {}^b v_q = {}^b\omega \times {}^b(\underbrace{\vec{r O_b}}_{-{}^b r})$$

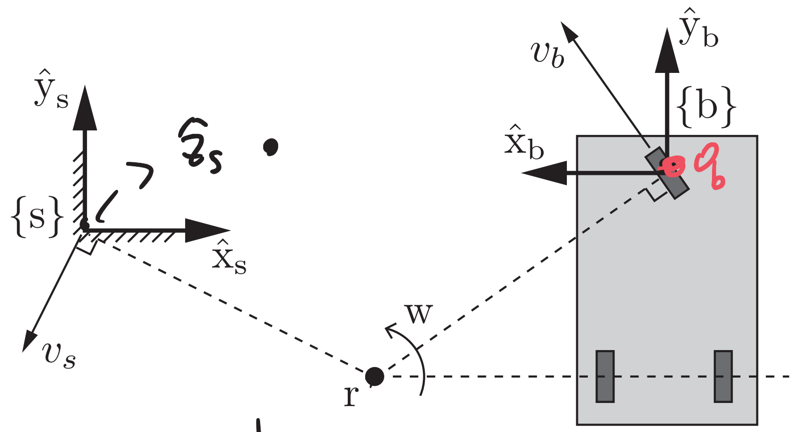
$$= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$${}^s\mathcal{V}_{car} = \begin{bmatrix} {}^s\omega \\ {}^s v_{O_s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ \\ \\ \end{bmatrix}$$

$${}^s v_{O_s} = {}^s\omega \times {}^s(\vec{r O_s}) \rightarrow -{}^s r$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$



${}^s r$ $\cancel{r_s} = (2, -1, 0)$, ${}^b r$ $\cancel{r_b} = (2, -1.4, 0)$, w=2 rad/s

Change Reference Frame for Twist (1/2)

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames $\{A\}$ and $\{B\}$

$$ {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix} $$

- They are related by ${}^A\mathcal{V} = {}^AX_B\,{}^B\mathcal{V}$

① ${}^A\omega = {}^AR_B\,{}^B\omega$

② "coordinate-free"

v_{O_A} : velocity of body-fixed pt currently coincides with O_A

v_{O_B} : - - - - - - - - - - - - with O_B

$$ {}^AT_B = \begin{bmatrix} {}^AR_B & {}^AO_B \\ 0 & 1 \end{bmatrix} $$

$\boxed{v_{O_A} = v_{O_B} + \omega \times \overrightarrow{O_BO_A}}$ $\xrightarrow{\text{choose } \{A\} \text{ frame to express "physics"}}$

$$ {}^Av_{O_A} = {}^Av_{O_B} + {}^A\omega \times {}^A(O_BO_A) $$
$$ = {}^AR_B\,{}^Bv_{O_B} + {}^AR_B\,{}^B\omega \times (-{}^AO_B) $$
$$ = {}^AR_B\,{}^Bv_{O_B} + {}^AO_B \times ({}^AR_B\,{}^B\omega) $$

Change Reference Frame for Twist (2/2)

$\Downarrow$

$[{}^A o_B]$

Combine ① and ②:

$${}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^A v_{o_A} \end{bmatrix} = \underbrace{\begin{bmatrix} {}^A R_B & 0 \\ [{}^A o_B]\,{}^A R_B & {}^A R_B \end{bmatrix}}_{6\times 6} \begin{bmatrix} {}^B\omega \\ {}^B v_{o_B} \end{bmatrix}$$

$$= \underbrace{\left[[{}^A o_B]\,{}^A R_B \quad {}^A R_B \right]}_{3\times 6} \underbrace{\begin{bmatrix} {}^B\omega \\ {}^B v_{o_B} \end{bmatrix}}_{{}^B\mathcal{V}}$$

$\triangleq {}^A X_B \leftrightarrow$ change of coordinate matrix for twist.

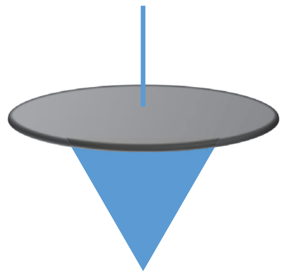
- If configuration {B} in {A} is $T = (R, p)$, then

$${}^A X_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Adjoint operator

Given $T = (R, p) \longrightarrow [\mathrm{Ad}_T] \Longrightarrow X$

Example I Revisited



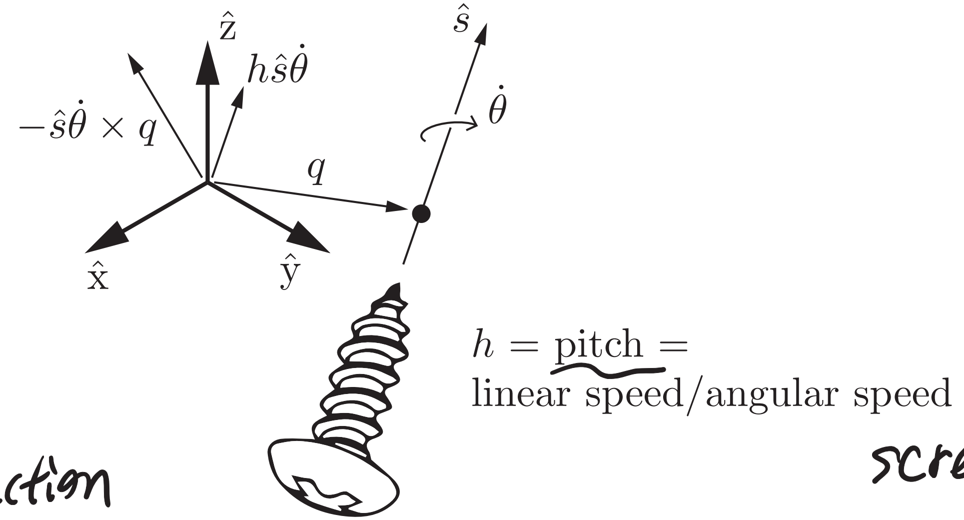
Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

- Recall: linear velocity: $v \in \mathbb{R}^3$ $\nearrow$, $v = \hat{v} \cdot \|v\|$.
  $\hat{v}$ → direction, $\|v\|$ → scalar
- angular velocity: $\omega \in \mathbb{R}^3$, $\omega = \hat{\omega}\dot{\theta}$ → scalar speed.
  $\hat{\omega}$ → direction
- rigid body velocity: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$ not directly direction vector ⊗ speed

Screw Motion: Definition

- Rotating about an axis while also translating along the axis



direction

screw motion: $(q, \hat{s}, h, \dot{\theta})$

- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

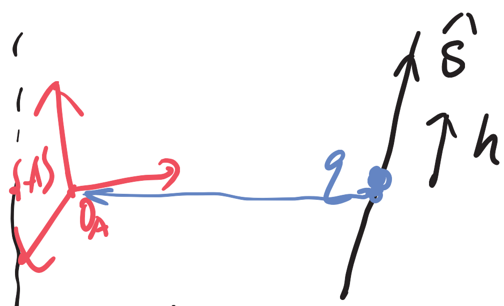
- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

From Screw Motion to Twist

screw motion is a special rigid body motion

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$

- Fix a reference frame $\{A\}$ with origin o_A.



- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^Av_{o_A})$

$${}^A\omega = {}^A\hat{s}\dot{\theta}$$

$${}^Av_{o_A} = {}^Av_q + {}^A\omega \times (-{}^Aq)$$

$$= {}^A\hat{s}(h\dot{\theta}) - {}^A\omega \times {}^Aq \quad \cdots \quad (*)$$

$${}^A\mathcal{V} = \text{ScrewToTwist}(\hat{s}, h, q, \dot{\theta})$$

"coordinate free": given arbitrary body-fixed point p. using q as reference

$$v_p = v_q + \omega \times \vec{qp}$$

$$\omega = \hat{s}\dot{\theta}$$

$$\hat{s}(h\dot{\theta})$$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$$\cancel{\omega = \hat{s}\dot{\theta} \qquad v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

$$\text{given } ({}^A\hat{s}, h, {}^Aq, \dot{\theta}) \xrightarrow{\text{eq. }(*)} {}^A\mathcal{V}$$

From Twist to Screw Motion

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\dot{\theta}$
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)

$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

  - If $\omega \neq 0$:

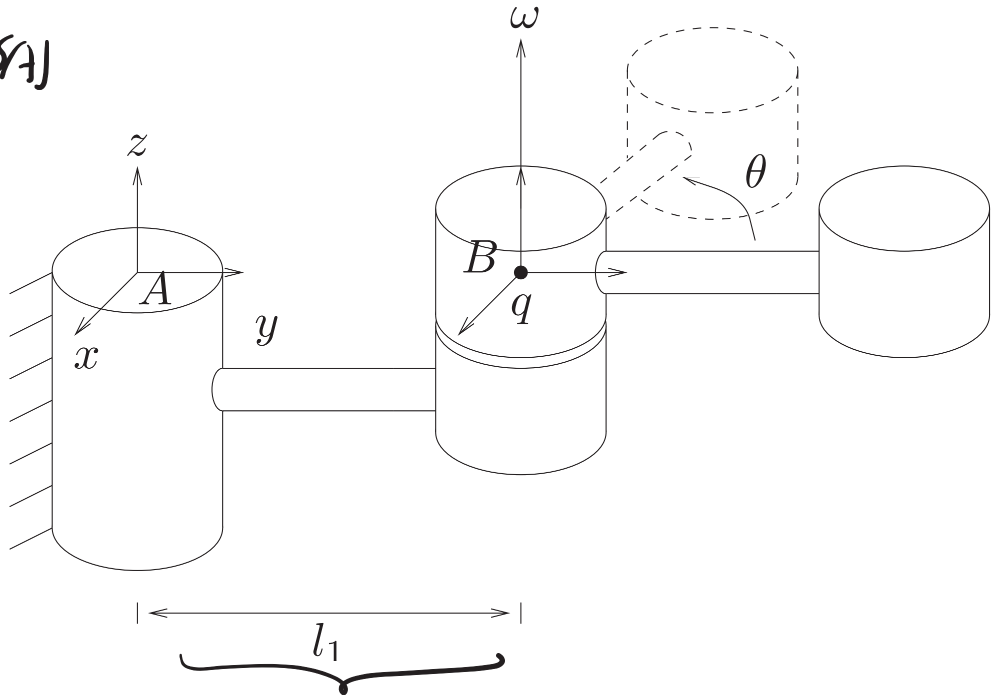
$$\underbrace{\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}}$$

you can plug into the eq. ⊛ to verify the result.

$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}$ = ScrewToTwist($\hat{s}$, $\dot{\theta}$, q, h)

Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$? chose {A}



· screw axis: ${}^A\hat{S} = {}^A\hat{z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^A q = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$

$\dot{\theta} = 2$, $\underline{h = 0}$

$${}^A\omega = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot 2}_{{}^A\hat{S} \cdot \dot{\theta}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$${}^A v_{o_A} = 0 - {}^A\omega \times {}^A q = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2l_1 \\ 0 \\ 0 \end{bmatrix}$$

- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

Screw Representation of a Twist

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

$\mathcal{V} = \hat{S}\dot{\theta}$

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

  - In this notation, we think of $\hat{\mathcal{S}}$ as the twist associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

$\hat{S} \longleftrightarrow (\hat{s}, h, q),\ \dot{\theta} = 1$

More Discussions

- linear motion velocity: $v = \underbrace{\|v\|} \cdot \hat{v}$ direction
- angular velocity $\omega = \hat{\omega} \cdot \dot{\theta}$
- Rigid body motion: (can be thought of as screw motion) $(\hat{s}, h, q, \dot{\theta})$
  - screw motion is a special rigid body motion, must has $\dot{\theta}$

    a twist: $\mathcal{V} = \text{ScrewToTwist}(\hat{s}, h, q, \dot{\theta}) = \text{ScrewToTwist}(\hat{s}, h, q, \dot{\theta}=1) \cdot \dot{\theta}$

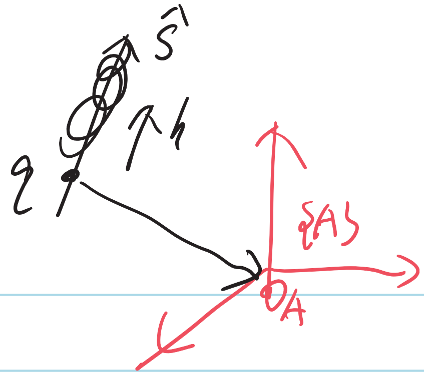
$$\begin{bmatrix} \omega \\ v_q \end{bmatrix} = \begin{bmatrix} \hat{s} \cdot \dot{\theta} \\ (h\dot{\theta})\hat{s} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{s} \\ h\hat{s} \end{bmatrix}} \dot{\theta}$$

twist corresponds to screw axis $(\hat{s}, h, q)$

unit speed screw motion

- $\mathcal{V} = S\dot{\theta}$

- given



$${}^{A}\mathcal{V} = \begin{bmatrix} {}^{A}\omega \\ {}^{A}v_{O_A} \end{bmatrix} \quad , \quad {}^{A}\omega = {}^{A}\hat{s}\cdot\dot{\theta}$$

$$(\dot{\theta})\,{}^{A}\hat{s}$$

$${}^{A}v_{O_A} = {}^{A}v_q + {}^{A}\omega \times {}^{A}(\overrightarrow{q O_A}) = (h\dot{\theta})\,{}^{A}\hat{s} + {}^{A}\omega \times (-{}^{A}q)$$

$$= \left(h\,{}^{A}\hat{s} - ({}^{A}\hat{s} \times {}^{A}q)\right)\dot{\theta}$$