MEE5114 Advanced Control for Robotics

Lecture 3: Operator View of Rigid-Body Transformation

Prof. Wei Zhang

SUSTech Insitute of Robotics
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.

- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n \times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric

- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω

- The set of skew-symmetric matrices in: $so(n) \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}$

$so(3)$

- We are interested in case $n = 2, 3$

Rotation matrix $R \in SO(3)$

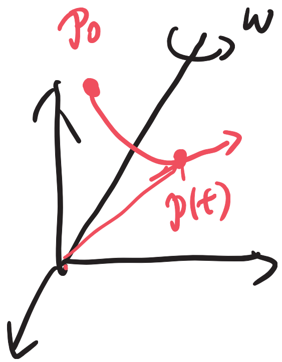
$\{R^T R = I, \det(R) = 1\}$

${}^A R_B = [{}^A x_B \ \ldots \ ]$

Rotation Operation via Differential Equation

- Consider a point initially located at p_0 at time $t = 0$

- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by



$$\dot{p}(t) = \hat{\omega} \times p(t) = [\hat{\omega}]p(t), \text{ with } p(0) = p_0 \tag{1}$$

linear velocity at time t

$OP(t)$

Linear ODE: $\dot{p}(t) = [\hat{w}]\, p(t)$

recall $\dot{x} = Ax$, $x(0) = x_0$

sln: $x(t) = e^{At} x_0$

$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$

- This is a linear ODE with solution: $\boxed{p(t) = e^{[\hat{\omega}]t} p_0}$.

- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta} p_0$

- $\mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

cordinate free

Rotation Matrix as a Rotation Operator (1/3)

Theorem:

- Every rotation matrix R can be written as $R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

Fact: any matrix of the form $e^{[\hat{\omega}]\theta}$ belongs $SO(3)$

Proof: $\left(e^{[\hat{\omega}]\theta}\right)^T (\cdot) = I$, $\Rightarrow \left(e^{[\hat{\omega}]\theta}\right)^T = \left(I + [\hat{\omega}]\theta + \frac{[\hat{\omega}]^2\theta^2}{2!} + \cdots\right)^T$

$$= \sum \frac{([\hat{\omega}]^T)^n \theta^n}{n!} = e^{-[\hat{\omega}]\theta}$$

$$e^A \cdot e^{-A} = I \Rightarrow \left(e^{[\hat{\omega}]\theta}\right)^T (\cdot) = I$$

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

- To apply the rotation operation, all the vectors/matrices have to be expressed in the **same reference frame** (this is clear from Eq (1))

Rotation Matrix as a Rotation Operator (2/3)

- For example, assume $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{Rot}(\hat{\mathrm{x}}; \pi/2)$
- Consider a relation $q = Rp$:
  - **Change reference frame interpretation** : two frames $\{A\}, \{B\}$, one physical point a
  - R : orientation of $\{B\}$ relative to $\{A\}$, i.e. $R = {}^A R_B$
  - then, p, q are coordinates of the same point a in $\{B\}$, $\{A\}$, respectively

    $p = {}^B a$, $q = {}^A a$ $\quad q = Rp \iff {}^A a = {}^A R_B \, {}^B a$
  - **Rotation operator interpretation**:

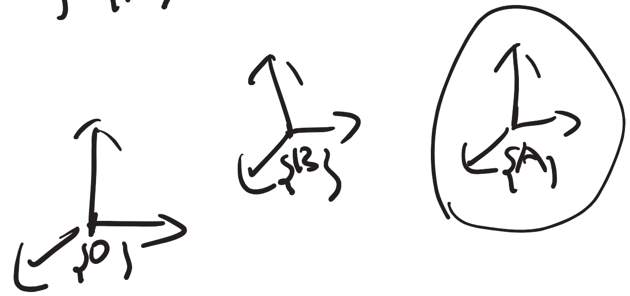
Have one frame $\{A\}$, and two points $a \xrightarrow{\mathrm{Rot}(\ )} a'$, $p = {}^A a$, $q = {}^A a'$

$${}^A a' = R \, {}^A a$$

Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame: $R_B = R R_A$

orientation of {A}



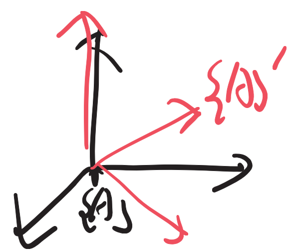
- Have "one frame object"

- Frame object {A}, orientation in {0}, ${}^0R_A$, in {B}, ${}^BR_A$

$${}^0R_A = {}^0R_B\, {}^BR_A$$

R

  - Rotating a frame: $R'_A = R R_A$



- two frames objects

- one refrence frame: {0}

$${}^0R_{A'} = R\, {}^0R_A$$

action

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

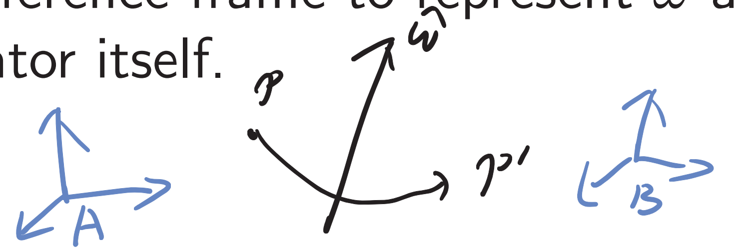
Rotation Matrix Properties

- $R^T R = I$ ← definition
- $R_1 R_2 \in SO(3)$, if $R_1, R_2 \in SO(3)$ product of rotation matrix is also rotation matrix
- $\|Rp - Rq\| = \|p - q\|$ ← rotation operator preserves distance

  $\|Rp - Rq\| = \|R(p-q)\|^2 = (p-q)^T \underbrace{R^T R}_{I} (p-q) = \|p-q\|$
- $\boxed{R(v \times w) = (Rv) \times (Rw)}$ ← rotation preserves orientation
- $R[w]R^T = [Rw]$

Rotation Operator in Different Frames (1/2)

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.



- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^{A}\hat{\omega}$ and {B}-frame coordinate ${}^{B}\hat{\omega}$. We know

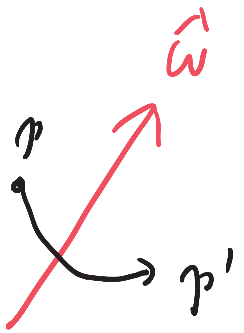
$$ {}^{A}\hat{\omega} = {}^{\cancel{B}\,A}R_{\cancel{A}\,B}\,{}^{B}\hat{\omega} $$

- Let ${}^{B}\mathrm{Rot}({}^{B}\hat{\omega}, \theta)$ and ${}^{A}\mathrm{Rot}({}^{A}\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

Rotation Operator in Different Frames (2/2)

- We have the relation:

$$^{A}\mathrm{Rot}(^{A}\hat{\omega},\theta) = {}^{A}R_{B}\,\underbrace{^{B}\mathrm{Rot}(^{B}\hat{\omega},\theta)}\,{}^{B}R_{A}$$



Approach 1: $p \xrightarrow{\mathrm{Rot}(\hat{\omega},\theta)} p' \underset{\{A\}}{\Rightarrow} {}^{A}p' = {}^{A}\mathrm{Rot}\,{}^{A}p$

$\overset{\{B\}\text{-frame}}{\Longrightarrow}$ ${}^{B}p' = {}^{B}\mathrm{Rot}\;{}^{B}p$

$\Rightarrow$ $\underbrace{{}^{A}R_{B}\,{}^{B}p'} = {}^{A}R_{B}\,{}^{B}\mathrm{Rot}\,{}^{B}p$ ${}^{B}R_{A}\,{}^{A}p$

${}^{A}p' = {}^{A}R_{B}\,{}^{B}\mathrm{Rot}\,{}^{B}R_{A}\;{}^{A}p$

$$\boxed{{}^{A}\mathrm{Rot} = \underbrace{{}^{A}R_{B}\,{}^{B}\mathrm{Rot}\,{}^{B}R_{A}}_{\text{similarity}}}$$

${}^{A}R_{B} = ({}^{B}R_{A})^{-1}$

Approach 2: Recall: $[Ra] = R[a]R^{T}$

A-frame Rotation:

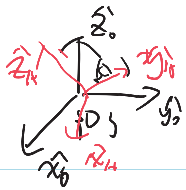
$${}^{A}\mathrm{Rot} = e^{[{}^{A}\hat{\omega}]\theta} = e^{[{}^{A}R_{B}{}^{B}\hat{\omega}]\theta}$$

$$= e^{{}^{A}R_{B}[{}^{B}\hat{\omega}]\,{}^{A}R_{B}^{-1}\theta}$$

$$= {}^{A}R_{B}\, e^{[{}^{B}\hat{\omega}]\theta}\, {}^{A}R_{B}^{-1}$$

$$= {}^{A}R_{B}\, {}^{B}\mathrm{Rot}\, {}^{A}R_{B}^{-1}$$

$$e^{PAP^{-1}} = Pe^{A}P^{-1}$$



{A} - orientation : ${}^0R_A$

example: Rotate {A}

about $\hat{x}_0$ by $\frac{\pi}{2}$

$${}^0\hat{x}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad {}^A\hat{x}_A = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$R = Rot(\hat{x}; \tfrac{\pi}{2}) = \underbrace{\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}}$$

, rotate about $\hat{x}$ axis of whichever frame we are working with

- $R \cdot {}^0R_A$:
- ${}^0R_A \cdot R$

$$R = e^{[\hat{x}]\frac{\pi}{2}}$$

$R \cdot {}^0R_A$: rotate {A} frame about $\hat{x}_0$ axis by $\frac{\pi}{2}$

$R \cdot {}^AR_B$: rotate {B} frame about $\hat{x}_A$ axis by $\frac{\pi}{2}$

$$R_A = [\hat{x}_A, \hat{y}_A, \hat{z}_A]$$

Operator: rotate about $\hat{x}_0$ axis by $\frac{\pi}{2}$

$$\underbrace{e^{[\hat{x}_0]\frac{\pi}{2}}}\, R_A \longrightarrow R_{A'}$$

choose {0} frame to express "physics"

$${}^0R_{A'} = e^{[{}^0\hat{x}_0]\frac{\pi}{2}} \cdot {}^0R_A$$

- What about rotating {A} about $\hat{x}_A$ axis

- $R_{A'} = e^{[\hat{x}_A]\frac{\pi}{2}} R_A$

$e^{[{}^A\hat{x}_A]\frac{\pi}{2}} \triangleq R$

use {s} frame to express physics:

$${}^0R_{A'} = e^{[{}^0\hat{x}_A]\frac{\pi}{2}}\, {}^0R_A$$

$$= e^{[{}^0R_A\, {}^A\hat{x}_A]\frac{\pi}{2}}\, {}^0R_A = ({}^0R_A\, R\, {}^0R_A^{-1})\, {}^0R_A$$

$$= {}^0R_A\, R$$

post-multiplication

$${}^0R_{A'} = {}^0R_A R$$

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Rigid-Body Operation via Differential Equation (1/3)

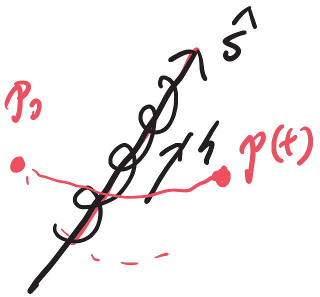
- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(1). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta)p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}]p(t)$ with $p(0) = p_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion

- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed ($\dot{\theta} = 1$). Let the corresponding twist be $\mathcal{V} = \mathcal{S} = (\omega, v)$. The motion can be described by the following ODE.

$p(t) \longrightarrow \tilde{p} = \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \quad \dot{\tilde{p}} = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}$



$$\boxed{\dot{p}(t) = \omega \times p(t) + v_r} \Rightarrow \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}\begin{matrix} 3\times 1 \\ 1\times 1 \end{matrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad (2)$$

$\dot{p}(t) = v_r + \omega \times \vec{r}_p(t)$

$\dot{\tilde{p}}(t) = \left[\begin{array}{c|c} - & - \end{array}\right] \cdot \tilde{p}(t) \Leftrightarrow \dot{x} = Ax$

- Solution to (2) in homogeneous coordinate is:

$$\underbrace{\begin{bmatrix} p(t) \\ 1 \end{bmatrix}}_{\tilde{p}(t)} = \boxed{\exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}}$$

Rigid-Body Operation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation

rigid body ve
spatial velocity

$$[\mathcal{V}] = \begin{array}{cc} 3\times 3 & 3\times 1 \\ \end{array}$$
$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$e^{[S]t} = I + \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t + \frac{(\quad)^2 t^2}{2!} + \cdots$$

$(\hat{s}, q, h), \dot{\theta} = 1$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$

$\mathcal{V} = S\dot{\theta}$

- With this notation, the solution to (2) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

$$e^{[S]t} = \begin{bmatrix} R & h \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \; h \in \mathbb{R}^3$$

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t

$se(3)$

$\forall\, w \in \mathbb{R}^3,\quad [w] \in so(3) \xrightarrow{\exp(\cdot)} e^{[w]} \in SO(3)$

$\forall\, S \in \mathbb{R}^6,\quad [S] = \begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix} \in se(3) \xrightarrow{\exp(\cdot)} e^{[S]} \in SE(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$\boxed{se(3)} = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

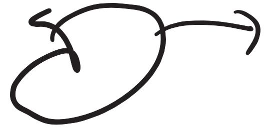
- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$



$\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \underline{\omega \times p} \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = \underline{e^{[\mathcal{V}]t}\tilde{p}(0)}$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree $e^{[S]\theta}$

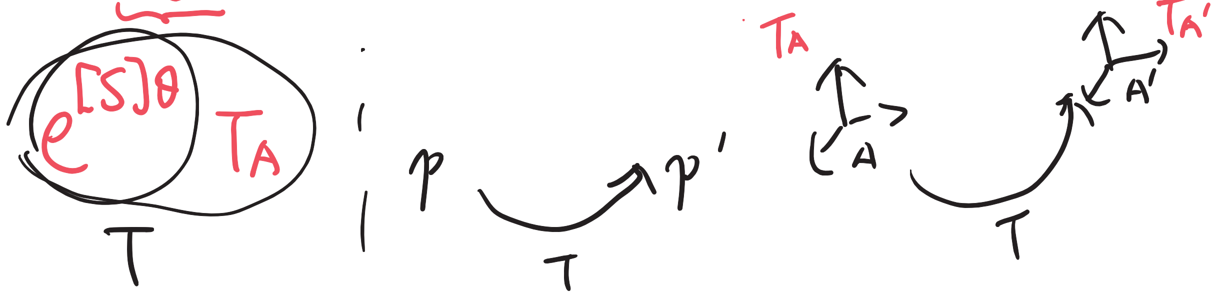
if not unit speed, $\mathcal{V} = S\dot{\theta}$

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree

$$\tilde{p}' = \underbrace{e^{[S]\theta}}_{T} \tilde{p},$$ using coordinate sys $\{o\}$ frame

${}^o\tilde{p}' = e^{[{}^oS]\theta}\, {}^o\tilde{p}$: (two points one frame)

- TT_A: "rotate" $\{A\}$-frame about $\mathcal{S}$ by θ degree



$${}^oT_{A'} = {}^oT\, {}^oT_A$$

For $T \in SE(3)$

- pose representation

${}^AT_B$: pose of $\{B\}$ relative to $\{A\}$

${}^A\tilde{p} = {}^AT_B\, {}^B\tilde{p}$: same point two frames

Rigid-Body Operator in Different Frames

- Expression of T in another frame (other than {O}):

$$\underset{\text{operation in }\{\mathrm{O}\}}{T} \quad \leftrightarrow \quad \underset{\text{operation in }\{\mathrm{B}\}}{T_B^{-1} T T_B}$$

Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

Rigid Operation on Screw Axis

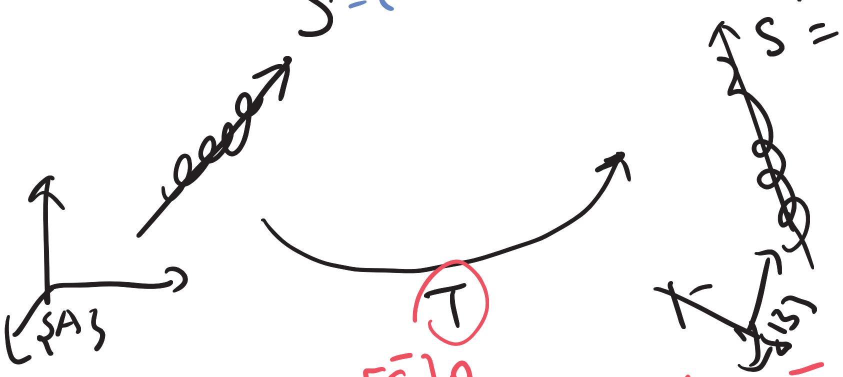
- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

proof:

$S = (\omega, v) \Leftrightarrow (\hat{s}, h, q)$

$S' = (\hat{s}', h', q') \Leftrightarrow (\omega', v')$



$e^{[\bar{S}]\theta}$: how does $\bar{S}$ relate to S, S'?

Not related at all

Let's work with an arbitrary frame {A} (rigidly attached to screw axis)

- Let {B} be the frame obtained by applying T operation $e^{[\bar{S}]\theta}$

- The coordinate of S in {A} is the same as the coordinate of ${}^{B}S'$ in {B}.

i.e. ${}^{A}S = {}^{B}S'$ ... (1)

More Space

- We also know $T = {}^AT_B$ ② (because: ${}^AT_B = T\,{}^AT_A$) $\quad I$

$T = {}^AT_B$

$${}^AT_B = \begin{bmatrix} {}^AR_B & {}^Ap_B \\ 0 & 1 \end{bmatrix}$$

$$= [\tilde{\hat{x}}_B, \tilde{\hat{y}}_B, \tilde{\hat{z}}_B, \tilde{p}_B]$$

- change of coordinate for twist

Multiply ${}^AX_B$ to both sides of ①

$${}^AX_B\,{}^AS = \underbrace{{}^AX_B\,{}^BS'}= {}^AS'$$

$$\Rightarrow \underbrace{{}^AS'}_{6\times1} = {}^AX_B\,{}^AS = \underbrace{[Ad_T]}_{6\times6}\,\underbrace{{}^AS}_{6\times1}$$

$$\| $$

$${}^AX_B = \begin{bmatrix} {}^AR_B & 0 \\ [{}^Ap_B]\,{}^AR_B & {}^AR_B \end{bmatrix} = [Ad_{{}^AT_B}] = [Ad_T]$$

$T = {}^AT_B$

screw axis S

rotate about $\bar{S}$ by θ

$$\underbrace{S'}_{6\times1} = \underbrace{[Ad_{e^{[S]\theta}}]}_{6\times6}\,\underbrace{S}_{6\times1}$$