MEE5114 Advanced Control for Robotics

Lecture 4: Exponential Coordinate of Rigid Body Configuration

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Outline

- Exponential Coordinate of $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of SE(3)

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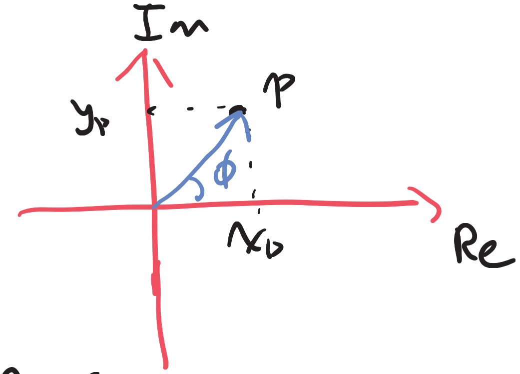
Towards Exponential Coordinate of $SO(3)$

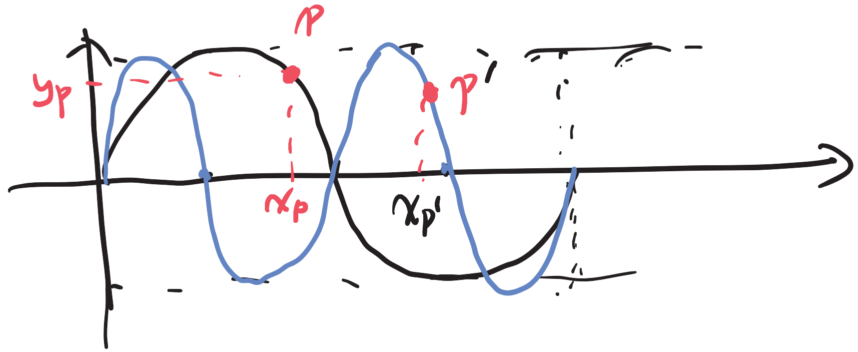


- Recall the polar coordinate system of the complex plane:
  - Every complex number $z = x + jy = \rho e^{j\phi}$
  - Cartesian coordinate $(x, y) \leftrightarrow$ polar coorindate (ρ, ϕ)

    $p = (x_p, y_p) \xrightarrow{\text{polar}} = (\rho_p, \phi_p)$
  - For some applications, polar coordinate is preferred due to its geometric meaning.

- Consider a set $M = \{\underbrace{(t, \sin(2n\pi t))}_{\in \mathbb{R}^2} : t \in (0, 1), n = 1, 2, 3, \ldots\}$

- $M \subseteq \mathbb{R}^2$
- $p \in M$; $p = (x_p, y_p)$



- Take advantage of structure of M

  Coordinate of p: $(1, x_p) \longleftrightarrow (x_p, \sin(2\pi x_p)) \in \mathbb{R}$

  p': $(2, x_{p'})$

Exponential Coordinate of $SO(3)$

- **Proposition** [Exponential Coordinate $\leftrightarrow$ SO(3)]
  - For any unit vector $[\hat{\omega}] \in so(3)$ and any $\theta \in \mathbb{R}$,

$$e^{[\hat{\omega}]\theta} \in SO(3)$$

  - For any $R \in SO(3)$, there exists $\hat{\omega} \in \mathbb{R}^3$ with $\|\hat{\omega}\| = 1$ and $\theta \in \mathbb{R}$ such that

$$R = e^{[\hat{\omega}]\theta}$$

$$\text{exp:} \quad [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3)$$

$$\text{log:} \quad R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3)$$

- The vector $\hat{\omega}\theta$ is called the *exponential coordinate* for R

- The exponential coordinates are also called the canonical coordinates of the rotation group $SO(3)$

Rotation Matrix as Forward Exponential Map

- Exponential Map: By definition $\omega = \hat{\omega}\dot{\theta}$ $\quad \dot{p} = \omega \times p = [\omega] p$

$$R \leftrightarrow e^{[\hat{\omega}]\theta} \triangleq I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots$$

- **Rodrigues' Formula**: Given any unit vector $[\hat{\omega}] \in so(3)$, we have

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

$A = [\omega]\theta$ → skew symmetric

Fact: if $\|\hat{\omega}\| = 1$, then $[\hat{\omega}] = -[\hat{\omega}]^T$

$[\hat{\omega}]^3 = -[\hat{\omega}]$

$[\hat{\omega}]^4 = [\hat{\omega}]^3[\hat{\omega}] = -[\hat{\omega}]^2$

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}([\hat{\omega}]^3) + \frac{\theta^4}{4!}([\hat{\omega}]^4) + \cdots$$

$$= I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta}[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \cdots\right)[\hat{\omega}]^2$$

($[\hat{\omega}]^3 \to -[\hat{\omega}]$, $\quad [\hat{\omega}]^4 \to -[\hat{\omega}]^2$)

Examples of Forward Exponential Map

$(1-\cos\theta)$

- Rotation matrix $R_x(\theta)$ (corresponding to $\hat{x}\theta$)

$Rot_x(\theta) \triangleq Rot(\hat{x};\theta) = e^{[\hat{x}]\theta} = I + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [\hat{x}] = [\;]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

- Rotation matrix corresponding to $(1, 0, 1)^T$

exp coordinate.

$\hat{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}, \quad \theta = \sqrt{2}$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow R = e^{[\hat{w}]\theta} = [\quad]$

Logarithm of Rotations

- If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and set $\hat{\omega}$ equal to one of the following

$$\frac{1}{\sqrt{2(1+r_{33})}}\begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{22})}}\begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{11})}}\begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

- Otherwise, $\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(R) - 1)\right) \in [0, \pi)$ and $[\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)$

Given any $R \in SO(3)$, $\Longrightarrow$ find $\hat{\omega}$ θ such that $e^{[\hat{\omega}]\theta} = R$

R ($\in \mathbb{R}^{3\times 3}$) $\xrightarrow{\log}$ $\hat{\omega}\theta$ ($\in \mathbb{R}^3$) $\leftarrow$ exp coordinate

R $\xleftarrow{e \text{ exp}}$ $\hat{\omega}\theta$

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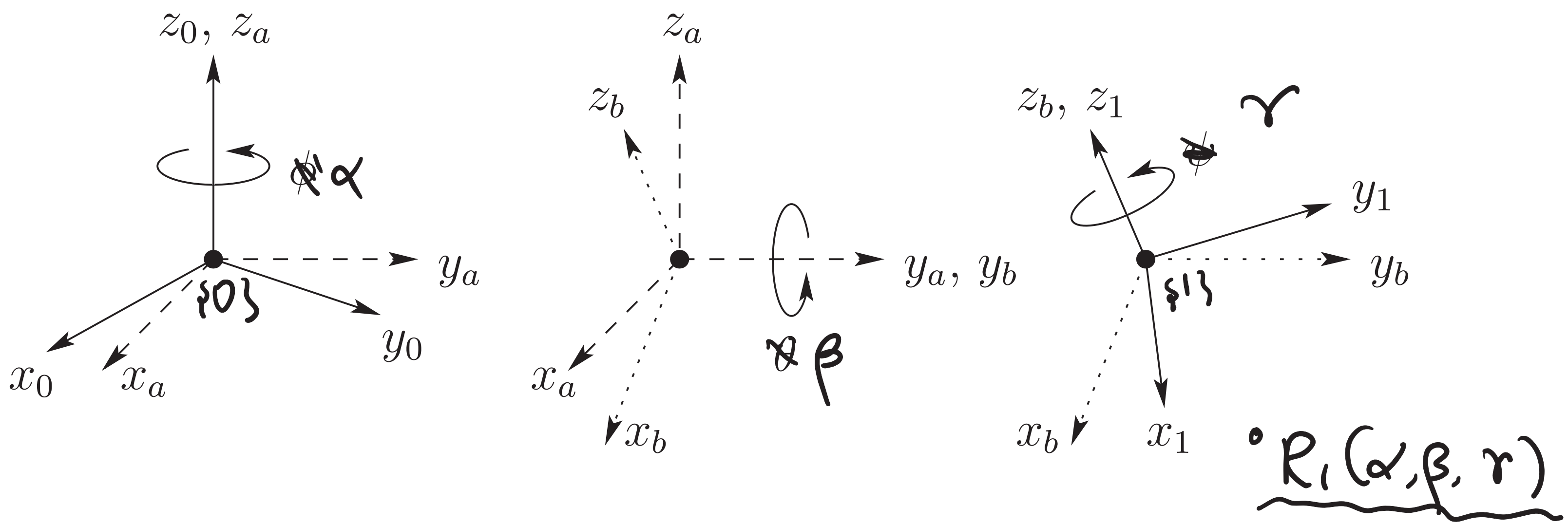
Euler Angle Representation of Rotation



- A common method of specifying a rotation matrix is through three independent quantities called **Euler Angles**.
- Euler angle representation
  - Initially, frame $\{0\}$ coincides with frame $\{1\}$
  - Rotate $\{1\}$ about $\hat{\mathrm{z}}_0$ by an angle α, then rotate about $\hat{\mathrm{y}}_\mathrm{a}$ axis by β, and then rotate about the $\hat{\mathrm{z}}_\mathrm{b}$ axis by γ. This yields a net orientation ${}^0R_1(\alpha, \beta, \gamma)$ parameterized by the ZYZ angles (α, β, γ)
  - ${}^0R_1(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

${}^0R_1(\alpha, \beta, \gamma) =$

step 1: ${}^0R_1(\alpha, 0, 0) = Rot(\hat{z}_0; \alpha)\, I$

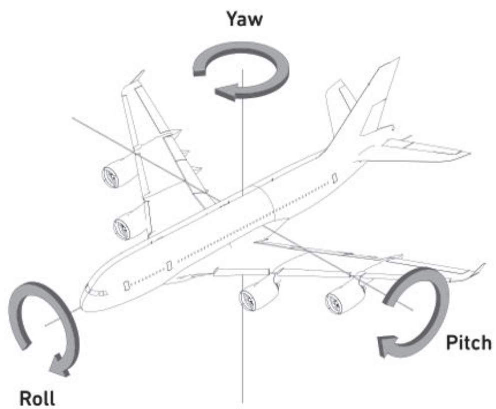
step 2: ${}^0R_1(\alpha, \beta, 0) = R(z_0; \alpha)\, R(\hat{y}; \beta)$

Other Euler-Like Parameterizations

- Other types of Euler angle parameterization can be devised using different ordered sets of rotation axes

roll-pitch-yaw :

- Common choices include:
  - ZYX Euler angles: also called *Fick angles* or yaw, pitch and roll angles
  - YZX Euler angles (Helmholtz angles)



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Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1 [Exponential Map of $se(3)$]: For any $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, we have $e^{[\mathcal{V}]\theta} \in SE(3)$ $= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- Case 1 ($\omega = 0$): $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- Case 2 ($\omega \neq 0$): without loss of generality assume $\|\omega\| = 1$. Then

$$e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{ with } G(\theta) = I\theta + (1-\cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2 \quad (1)$$

For any twist $\mathcal{V}, \theta \longrightarrow e^{[\mathcal{V}]\theta} \in SE(3)$

Log of $SE(3)$: from Rigid-Body Motion to Twist

Theorem 2 [Log of $SE(3)$]: Given any $T = (R, p) \in SE(3)$, one can always find twist $\mathcal{S} = (\omega, v)$ and a scalar θ such that

unit velocity along screw axis

$\mathcal{S} \Leftrightarrow (\hat{s}, h, q), \dot{\theta} = 1$

$$e^{[\mathcal{S}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm Algorithm:

- If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.
- Otherwise, use matrix logarithm on $SO(3)$ to determine ω and θ from R. Then v is calculated as $v = G^{-1}(\theta)p$, where

$$G^{-1}(\theta) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2}\right)[\omega]^2$$

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \Rightarrow [\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \xrightarrow{\exp(\cdot)} e^{[\mathcal{V}]\theta} \in SE(3)$$

$[\mathcal{V}] \in se(3)$

Exponential Coordinates of Rigid Transformation

- To sum up, screw axis $\mathcal{S} = (\omega, v)$ can be expressed as a normalized twist; its matrix representation is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

- A point started at $p(0)$ at time zero, travel along screw axis $\mathcal{S}$ at unit speed for time t will end up at $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$

- Given $\mathcal{S}$ we can use Theorem 1 to compute $e^{[\mathcal{S}]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $\mathcal{S} = (\omega, v)$ and θ such that $e^{[\mathcal{S}]\theta} = T$.

- We call $\mathcal{S}\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$

More Space

$\mathcal{V} \cdot S$ $\qquad$ $e^{[w]} \in SO(3)$

$w = \hat{w}\dot{\theta}$ $\qquad$ $e^{[\hat{w}]\theta}$

Given $T \in SE(3)$, $\Longrightarrow$ $\mathcal{V}$ such $e^{[\mathcal{V}]} = T$

More Space