$[\omega] = \dot{R}R^{-1}$ $[\mathcal{V}] = \dot{T}T^{-1}$

MEE5114 Advanced Control for Robotics

Lecture 5: Instantaneous Velocity of Moving Frames

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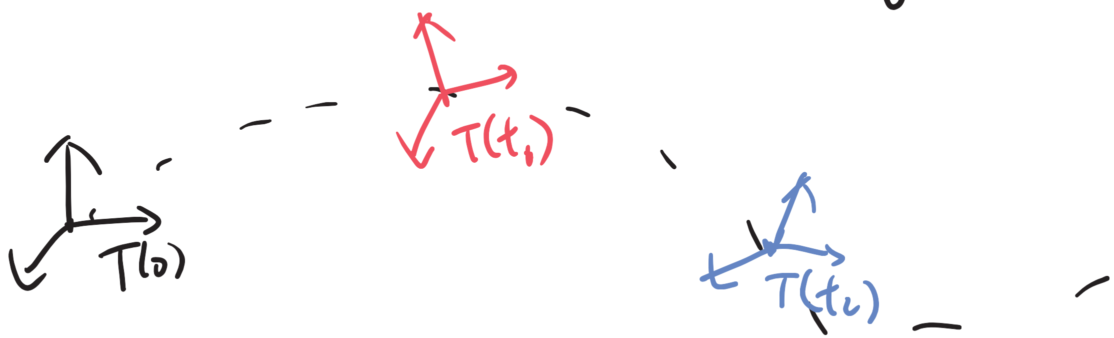
Outline

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

Objectives: Given frame trajectory $T(t) = (R(t), p(t))$



Question: what is the velocity of $T(t)$ at time t

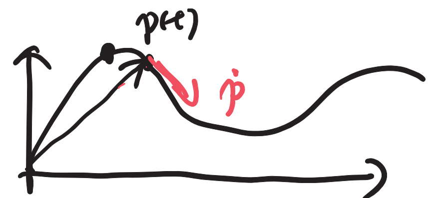
twist / spatial velocity $\in \mathbb{R}^6$

Outline

- $T(t) \in \mathbb{R}^{4\times 4}$: 4×4 matrix in $SE(3)$
- $\log(T(t)) \to S\theta$: is $S\theta$ the velocity of $T(t)$?

NO

e.g.



- Instantaneous Velocity of Rotating Frames

· $p(t)$: "position" vector $\longleftrightarrow$ $S\theta \Leftrightarrow T(t)$ "position"

· $\dot{p}(t)$: velocity vector $\longleftrightarrow$?

- Instantaneous Velocity of Moving Frames

- $\dot{T}(t)$ is velocity of $T(t)$?

NO

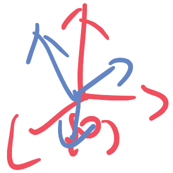
- How about $\log(\dot{T}(t))$

$\notin SE(3)$

Instantaneous Velocity of Rotating Frame (1/2)

- {A} frame is rotating with orientation $R_A(t)$ and velocity $\omega_A(t)$ at time t (Note: everything is wrt {O}-frame)

$\hat{\omega}\theta \neq \omega_A$



- Let $\hat{\omega}\theta = \log(R_A(t))$ be its exp. coordinate. ⟵ "position" not velocity
  - Note: $\hat{\omega}\theta$ means $R_A(t)$ can be obtained from the reference frame (say {O}-frame) by rotating about $\hat{\omega}$ by θ degree.

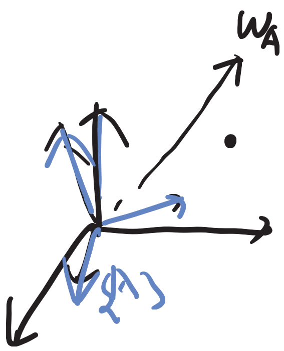
$\hat{\omega}\theta$

  - $\hat{\omega}\theta$ only describes the current orientation of {A} relative to {O}, it does not contain info about how the frame is rotating at time t.

Instantaneous Velocity of Rotating Frame (2/2)

- What is the relation between $\omega_A(t)$ and $R_A(t)$?

$$\frac{d}{dt}R_A(t) = [\omega_A(t)]R_A(t) \Rightarrow [\omega_A(t)] = \dot{R}_A(t)R_A^{-1}(t)$$



$$R_A(t) = [\hat{x}_A, \hat{y}_A, \hat{z}_A]$$

$$\dot{\hat{x}}_A = \omega_A \times \hat{x}_A, \quad \dot{\hat{y}}_A = \omega_A \times \hat{y}_A, \quad \dot{\hat{z}}_A = \omega_A \times \hat{z}_A$$

$$\dot{R}_A = [\dot{\hat{x}}_A \ \dot{\hat{y}}_A \ \dot{\hat{z}}_A] = \omega_A \times R_A = [\omega_A]R_A$$

$${}^o\dot{R}_A = [{}^o\omega_A]\,{}^oR_A \Longrightarrow [{}^o\omega_A] = {}^o\dot{R}_A\,{}^oR_A^{-1}$$

- Question 2: ${}^A\omega_A = ?$ $\quad {}^A\omega_A = {}^AR_o\,{}^o\omega_A$

$$[{}^o\omega_A] = {}^o\dot{R}_A\,{}^oR_A^{-1} \Longrightarrow [{}^A\omega_A] = [{}^AR_o\,{}^o\omega_A] = {}^AR_o[{}^o\omega_A]\,{}^AR_o^T$$

$$= {}^AR_o\,{}^o\dot{R}_A\,\cancel{{}^oR_A^{-1}}\,\cancel{{}^oR_A}$$

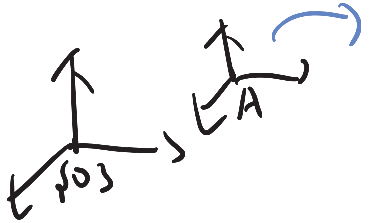
$$[{}^A\omega_A] = {}^oR_A^{-1}\,{}^o\dot{R}_A \qquad = {}^oR_A^{-1}\,{}^o\dot{R}_A$$

Outline

- Instantaneous Velocity of Rotating Frames
- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Moving Frame (1/2)

- {A} moving frame with configuration $T_A(t)$ at time t undergoes a rigid body motion with velocity $\mathcal{V}_A(t) = (\omega, v)$ (Note: everything is wrt {O}-frame)



$T_A(t) = (R_A(t), p_A(t))$

[$\quad e^{[\hat{S}]\theta} = T_A$

- The exponential (pose) coordinate $\hat{S}\theta = \log(T_A(t))$ only indicates the current configuration of {A}, and does not tell us about how the frame is moving at time t.

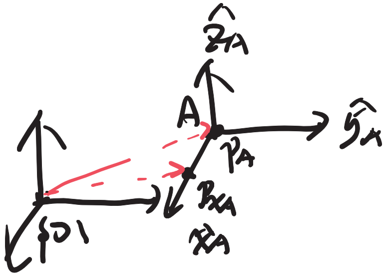
- $\hat{S}\theta$ is exponential coordinate of T_A

- $T_A(t)$ changes with time, we want to know A's spatial velocity

Instantaneous Velocity of Moving Frame (2/2)

- What is the relation between $\mathcal{V}_A(t)$ and $T_A(t)$?

$$\frac{d}{dt}T_A(t) = [\mathcal{V}_A(t)]T_A(t) \Rightarrow [\mathcal{V}_A(t)] = \dot{T}_A(t)T_A^{-1}(t)$$



- $T_A(t) = \begin{bmatrix} \hat{x}_A & \hat{y}_A & \hat{z}_A & p_A \\ 0 & 0 & 0 & 1 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{\hat{x}}_A & \tilde{\hat{y}}_A & \tilde{\hat{z}}_A & \tilde{p}_A \end{bmatrix}$

homogeneous coordinate of axes.

Given: - current pose $T_A(t)$ as func of time

- suppose {A} velocity is $\mathcal{V}_A = \begin{bmatrix} \omega \\ v_r \end{bmatrix}$

any body-fixed point q, $v_q = v_r + \omega \times \vec{rq}$

- $\dot{T}_A = \begin{bmatrix} \dot{\tilde{\hat{x}}}_A & \dot{\tilde{\hat{y}}}_A & \dot{\tilde{\hat{z}}}_A & \dot{\tilde{p}}_A \end{bmatrix}$

- $\dot{\tilde{\hat{x}}}_A = \begin{bmatrix} \dot{\hat{x}}_A \\ 0 \end{bmatrix} \rightarrow \underline{\dot{\hat{x}}_A \overset{?}{=} \omega \times \hat{x}_A}$ ✓

$\hat{x}_A = p_{x_A} - p_A$

$\dot{\hat{x}}_A = \dot{p}_{x_A} - \dot{p}_A = \cancel{v_r} + \omega \times \vec{r p_{x_A}} - \cancel{v_r} - \omega \times \vec{r p_A}$

$= \omega \times (p_{x_A} - p_A) = \boxed{\omega \times \hat{x}_A}$

More Space

$$\dot{\tilde{\hat{x}}}_A = \begin{bmatrix} \omega \times \hat{x}_A \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} [\omega] & v_r \\ 0 & 0 \end{bmatrix}}_{[\mathcal{V}_A]} \underbrace{\begin{pmatrix} \hat{x}_A \\ 0 \end{pmatrix}}_{\tilde{\hat{x}}_A}$$

$$\dot{\tilde{\hat{x}}}_A = [\mathcal{V}_A] \tilde{\hat{x}}_A \quad \cdots \quad (1)$$

Similarly, $\dot{\tilde{\hat{y}}}_A = [\mathcal{V}_A] \tilde{\hat{y}}_A \quad \cdots \quad (2)$, $\dot{\tilde{\hat{z}}}_A = [\mathcal{V}_A] \tilde{\hat{z}}_A \quad \cdots \quad (3)$

$$\dot{\tilde{p}}_A = \begin{bmatrix} \dot{p}_A \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v_r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_A \\ 1 \end{bmatrix} = [\mathcal{V}_A] \tilde{p}_A$$

Note: $\dot{p}_A = v_r + \omega \times \vec{r p_A}$, choose r to be $\{o\}$ origin

$$= v_r + \omega \times p_A$$

$${}^o\dot{T}_A = [{}^o\mathcal{V}_A]\, {}^oT_A \Rightarrow [{}^o\mathcal{V}_A] = {}^o\dot{T}_A\, {}^oT_A^{-1} \qquad {}^A\mathcal{V}_A = {}^AX_o\, {}^o\mathcal{V}_A$$

similar to the rotation case: $[{}^A\mathcal{V}_A] = {}^oT_A^{-1}\, {}^o\dot{T}_A$