PoE

MEE5114 Advanced Control for Robotics

# Lecture 6: Product of Exponential and Kinematics of Open Chain

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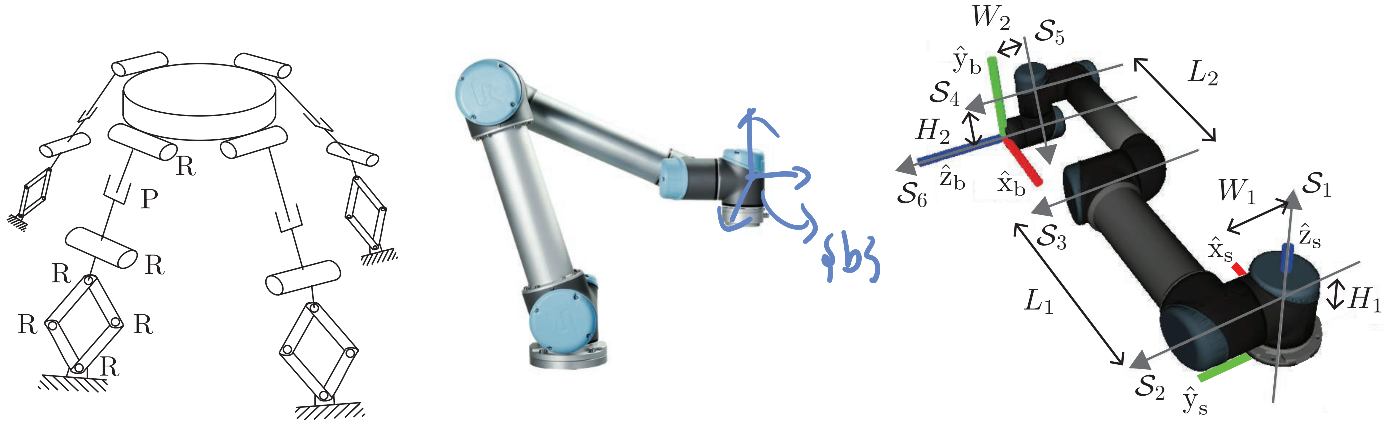
# Outline

- Motivating Example
- Product of Exponential Formula Derivations
- Practice Example

# Kinematics : Robot : Multiple rigid bodies interconnected through joints.

**Kinematics** is a branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the mass of each or the forces that caused the motion




pose

- **Forward Kinematics**: calculation of the configuration $T = (R, p)$ of the end-effector frame from joint variables $\theta = (\theta_1, \ldots, \theta_n)$

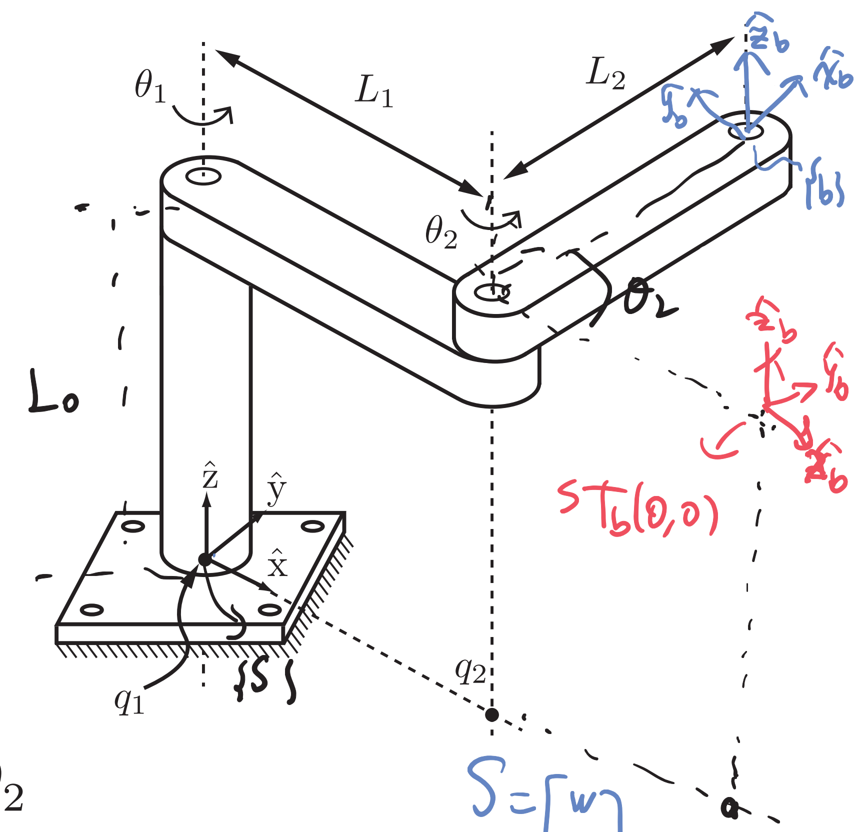
- **Velocity Kinematics (Next Lecture)**: Deriving the Jacobian matrix: linearized map from the joint velocities $\dot{\theta}$ to the spatial velocity $\mathcal{V}$ of the end-effector

# Illustration Example (1/3)

$^sT_b(\theta_1, \theta_2)$

Consider a 2R robot

- Three links and two joints $\theta_1, \theta_2$
- Link/body frame attached to link $i$ at joint $i$ (one of possible choices)
- Fixed/world frame {s} frame , end-effector frame {b}




- **Goal**: compute $^sT_b(\theta_1, \theta_2)$: function of $\theta_1, \theta_2$

$S = \begin{bmatrix} w \\ v \end{bmatrix}$

Home pose

- Initial pose: $M \triangleq {}^sT_b(0,0)$

$S = (\hat{s}, h, q_2)$

$$M = {}^sT_b(0,0) = \begin{bmatrix} 1 & 0 & 0 & L_1+L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Illustration Example (2/3)

- Fix joint 1 at $\theta_1 = 0$, rotate joint 2 by $\theta_2$, we have ${}^sT_b(0, \theta_2)$

- Rigid body motion for Link/body 2 ({b}), represented by screw motion with screw axis $S_2$. In coordinate free way.

  Initial: $\underset{\theta_1=0,\ \theta_2=0}{M} \xrightarrow{\hat{T} = e^{[S_2]\theta_2}} \hat{T}M = e^{[S_2]\theta_2} M$

- In {s}-frame : ${}^sT_b(0,\theta_2) = e^{[{}^0S_2]\theta_2}\, {}^sT_b(0,0)$

  {s} = {0}

  - ${}^0S_2$ is a function of $\theta_1$, more precisely, should be written as ${}^0S_2(\theta_1)$
  - For current step $\theta_1 = 0$, define ${}^0\bar{S}_2 \triangleq {}^0S_2(0)$

  - ${}^0\bar{S}_2 = \begin{bmatrix} {}^0w_2 \\ {}^0v_2 \end{bmatrix}$, ${}^0w_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^0v_2 = \cancel{h\,{}^0s} - {}^0w_2 \times {}^0q_2 = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$

  $\Rightarrow {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$

# Illustration Example (3/3)

- Fix joint 2 at $\theta_2$, and rotate joint 1 by $\theta_1 \Rightarrow {}^sT_b(\theta_1, \theta_2)$

- joint 1 screw axis $S_1$, $S_1$ : independent of $\theta_1, \theta_2$, ${}^0\bar{S}_1 = {}^0S_1(\theta_1=0, \theta_2=0)$

$$ {}^0\bar{S}_1 = \begin{bmatrix} {}^0w_1 \\ {}^0v_1 \end{bmatrix}, \quad {}^0w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}^0v_1 = h\,{}^0\hat{S}_1 - {}^0\hat{S}_1 \times {}^0q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} $$

- Initial : $\underline{{}^sT_b(0, \theta_2)} \xmapsto{\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1}} {}^sT_b(\theta_1, \theta_2) = e^{[{}^0\bar{S}_1]\theta_1}\, {}^sT_b(0, \theta_2)$

$$ \Rightarrow {}^sT_b(\theta_1, \theta_2) = \underbrace{e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} M}_{4\times 4} $$

PoE

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# Notation Setup (1/2)

- Suppose that the robot has $n$ joints and $n$ links. Each joint has one degree of freedom represented by joint variable $\theta_i$, $i = 1, \ldots, n$
  - $\theta_i$: the joint angle (Revolute joint) or joint displacement (Primatic joint)

{0}

- Specify a fixed frame {s}: also referred to as frame {0}

frame {i} moves with body i

- Attach frame $\{i\}$ to link $i$ at joint $i$, for $i = 1, \ldots, n$
- Attach frame {b} at the end-effector: sometimes referred to as frame {n+1}

Constant

- ${}^i\mathcal{S}_i$: screw axis of joint $i$ expressed in frame {i}
- ${}^0\mathcal{S}_i$: screw axis of joint $i$ expressed in fixed frame {0} (i.e. frame {s})

${}^0S_i(\theta_1, \theta_2, \cdots, \theta_i)$

# Notation Setup (2/2)

${}^{s}T_b$

- For simplicity, we write configuration as $T_{sb}$, which is the same as ${}^{s}T_b$. Similarly, $T_{ij} = {}^{i}T_j$

- Note: ${}^{i}\mathcal{S}_i$ does not change when the robot moves (i.e. when $\theta$ changes), but ${}^{0}\mathcal{S}_i$ depends on $\theta_1, \ldots, \theta_i$. Sometimes, we write out the dependency explicitly, i.e. ${}^{0}\mathcal{S}_i(\theta_1, \ldots, \theta_i)$

- Define home position: $\theta_1 = 0, \ldots, \theta_n = 0$. This is the configuration when all the joint angles are zero. One can also choose other *fixed* angles as the home position

constant by definition

- Define ${}^{0}\bar{\mathcal{S}}_i = {}^{0}\mathcal{S}_i(0, \ldots, 0)$: the screw axis of joint $i$ expressed in frame $\{0\}$, when the robot is at the home position.

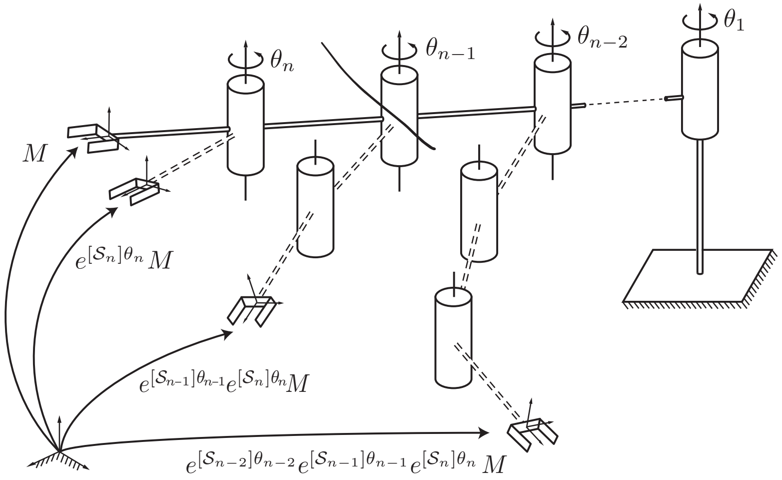
# Product of Exponential: Main Idea

${}^{s}T_{b}(\theta_1, \theta_2 \cdots \theta_n)$

- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$

step 0 :

- Compute $M \triangleq T_{sb}(0, \ldots, 0)$: the configuration of end-effector when the robot is at home position   Let $\theta_1 = \theta_2 \cdots = \theta_n = 0$




- Apply screw motion to joint $n$: $T_{sb}(0, \ldots, 0, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$   this screw motion does not changes $S_{n-1}, S_{n-2} \cdots$ screw axes

$e^{[{}^0\bar{S}_{n-1}]\theta_{n-1}}$

- Apply screw motion to joint $n-1$ to obtain:

$$T_{sb}(0, \ldots, 0, \theta_{n-1}, \theta_n) = e^{[{}^0\bar{\mathcal{S}}_{n-1}]\theta_{n-1}} e^{[{}^0\bar{\mathcal{S}}_n]\theta_n} M$$

- After $n$ screw motions, the overall forward kinematics:

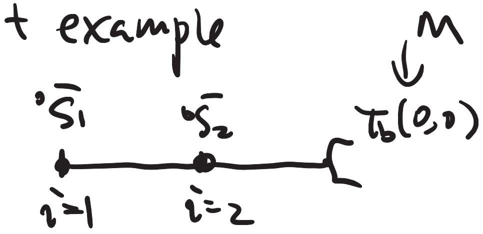
$$\boxed{T_{sb}(\theta_1, \ldots, \theta_n) = \underbrace{e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} e^{[{}^0\bar{\mathcal{S}}_2]\theta_2} \cdots e^{[{}^0\bar{\mathcal{S}}_n]\theta_n}}_{\text{PoE}} M}$$

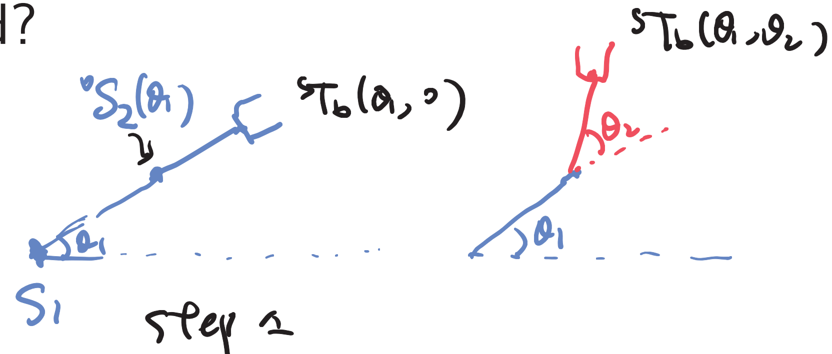
# PoE: Screw Motions in Different Order (1/2)

- PoE was obtained by applying screw motions along screw axes ${}^0\bar{\mathcal{S}}_n$, ${}^0\bar{\mathcal{S}}_{n-1}$, .... What happens if the order is changed?

to go back to first example

Top view




- For simplicity, assume that $n = 2$, and let us apply screw motion along ${}^0\bar{\mathcal{S}}_1$ first:
  - $T_{sb}(\theta_1, 0) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} M$
  - Now screw axis for joint 2 has been changed. The new axis ${}^0\mathcal{S}_2 = {}^0\mathcal{S}_2(\theta_1, 0) \neq {}^0\bar{\mathcal{S}}_2$.

$${}^sT_b(\theta_1,\theta_2) = e^{[{}^0S_2(\theta_1,0)]\theta_2} \cdot e^{[{}^0\bar S_1]\theta_1} M$$

$\neq {}^0\bar S_2$

Initial

$${}^0\bar S_2 \xrightarrow{\hat T_1 = e^{[{}^0\bar S_1]\theta_2}} \underbrace{{}^0S_2(\theta_1)}_{6\times 1} = \underbrace{[Ad_{\hat T_1}]}_{6\times 6} \underbrace{{}^0\bar S_2}_{6\times 1}$$

# PoE: Screw Motions in Different Order (2/2)

- $T_{sb}(\theta_1,\theta_2) = e^{[{}^0\mathcal{S}_2]\theta_2} T_{sb}(\theta_1, 0)$

$$[[Ad_T] S]$$

$$\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- Recall: $[\underbrace{R\omega}_{3\times 1}] = R[\omega]R^{-1}$

Fact: If $S' = \underbrace{[Ad_T] S}$ $\Longleftrightarrow$ $[S'] = T \underbrace{[S]}_{4\times 4} T^{-1}$

- Based on this fact:

$$e^{[{}^0S_2(\theta_1)]\theta_2} = e^{[[Ad_{\hat{T}_1}]{}^0\bar{S}_2]\theta_2} = e^{\hat{T}_1[{}^0\bar{S}_2]\hat{T}_1^{-1}\theta_2}$$

$$= \underbrace{\hat{T}_1 e^{[{}^0\bar{S}_2]\theta_2} \hat{T}_1^{-1}} \quad \cdots$$

$$e^{PAP^{-1}} = P e^{A} P^{-1}$$

plug in: ${}^sT_b(\theta_1,\theta_2) = \hat{T}_1 e^{[{}^0\bar{S}_2]\theta_2} \underbrace{\cancel{\hat{T}_1^{-1}} \cancel{e^{[{}^0\bar{S}_1]\theta_1}}} M$

Note: $\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1}$

$$= e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2} M$$

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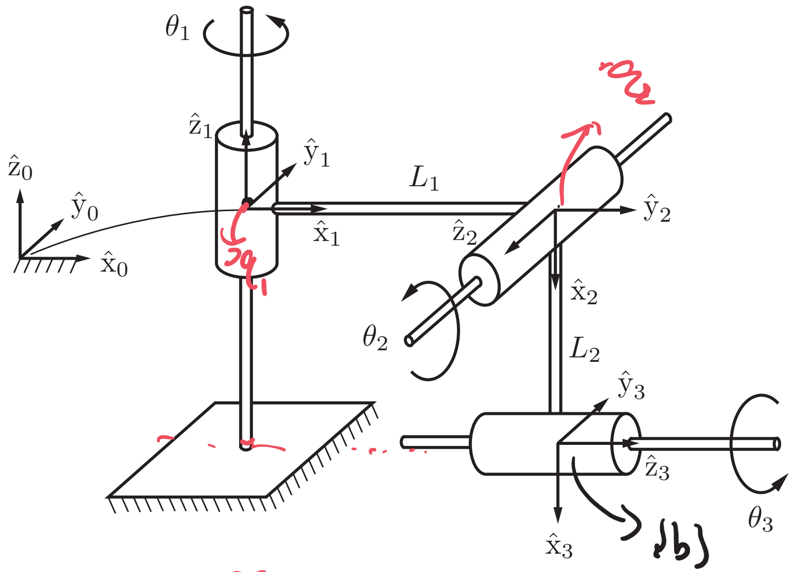
# PoE Example: 3R Spatial Open Chain




$\{b\} = \{3\}$

- Find ${}^0T_b(\theta_1, \theta_2, \theta_3)$

• Step 1: Initial pose $M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$

• step 2: ${}^0\bar{S}_1 = ({}^0\omega_1, {}^0v_1)$, ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ${}^0v_1 = 0 - {}^0\omega_1 \times {}^0q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0\bar{S}_2 = ({}^0\omega_2, {}^0v_2)$, ${}^0\omega_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, pick ${}^0q_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$, ${}^0v_1 = -\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{bmatrix}$

${}^0\bar{S}_3 = ({}^0\omega_3, {}^0v_3)$, ${}^0\omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, ${}^0q_3 = \begin{bmatrix} L_1 \\ 0 \\ -L_2 \end{bmatrix}$

# More Discussions

$$ {}^{0}v_3 = -{}^{0}\omega_3 \times {}^{0}q_3 = \begin{bmatrix} \times \\ \times \\ \times \end{bmatrix} $$

- 

$$ \Rightarrow \quad {}^{s}T_b(\theta_1, \theta_2, \theta_3) = e^{[{}^{0}\bar{S}_1]\theta_1} e^{[{}^{0}\bar{S}_2]\theta_2} e^{[{}^{0}\bar{S}_3]\theta_3} M \quad \checkmark $$