MEE5114 Advanced Control for Robotics

Lecture 1: Linear Differential Equations and Matrix Exponential

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Outline

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

Motivations

- Most engineering systems (including most robotic systems) are modeled by Ordinary Differential (or Difference) Equations (ODEs) / or PDE

2nd order

differential equ in variable θ

- Example: Dynamics of 2R robot

$$\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta,\dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})},$$

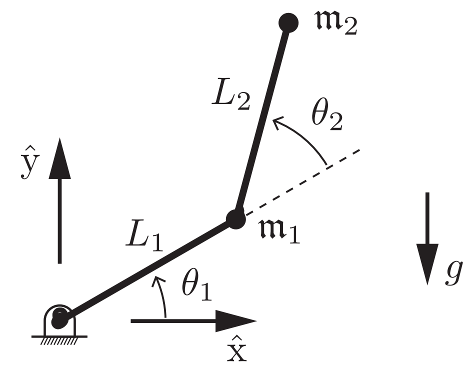
τ: input

with

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix},$$



- Screw theory, exponential coordinate, and Product of Exponential (PoE) are based on the (linear) differential equation view of robot kinematics

Linear Differential Equations (Autonomous)

- Linear Differential Equations: ODEs that are linear wrt variables e.g.:

① $\begin{cases} \dot{x}_1(t) + x_2(t) = 0 \\ \dot{x}_2(t) + x_1(t) + x_2(t) = 0 \end{cases}$; ② $\begin{cases} \ddot{y}(t) + z(t) = 0 \\ \dot{z}(t) + y(t) = 0 \end{cases}$

- two coupled 1^{st}-order ODE involve x_1, x_2

vector form: $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ in $\mathbb{R}^2$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\,x$$

. 2^{nd}-order ODE in 2 variables z, y

$\Rightarrow$ vector state space form:

$x_1 = y$. $x_2 = \dot{y}$, $x_3 = z$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

- State-space form (1st-order ODE with vector variables):

Linear $\dot{x} = Ax$

General Linear Control Systems

All ODEs can be written in state-space form

1st order ODE in vector space

if Linear: $f(x) = A \cdot x$

- General (Autonomous) Dynamical Systems: $\dot{x}(t) = f(x(t))$
  - $x(t) \in \mathbb{R}^n$: state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$: *vector field*

"Autonomous" means "f" depends only on "x".

- Non-autonomous: $\dot{x}(t) = f(x(t), t)$ ← captures all "non-x" dependence.

e.g. $\dot{x} = Ax + \sin(t)$, $\dot{x} = Ax + d(t)$

- Control Systems: $\dot{x}(t) = \boxed{f(x(t), u(t))}$
  - vector field $f : \mathbb{R}^n \times \mathbb{R}^m$ depends on external variable $u(t) \in \mathbb{R}^m$

$\dot{x} = Ax + \sin(u)$

vector field: $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$f(q, w) = Aq + Bw$

- General Linear Control Systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

← static relation

$f(x, u) = Ax + Bu$

$f(x(t), u(t)) = Ax(t) + Bu(t)$

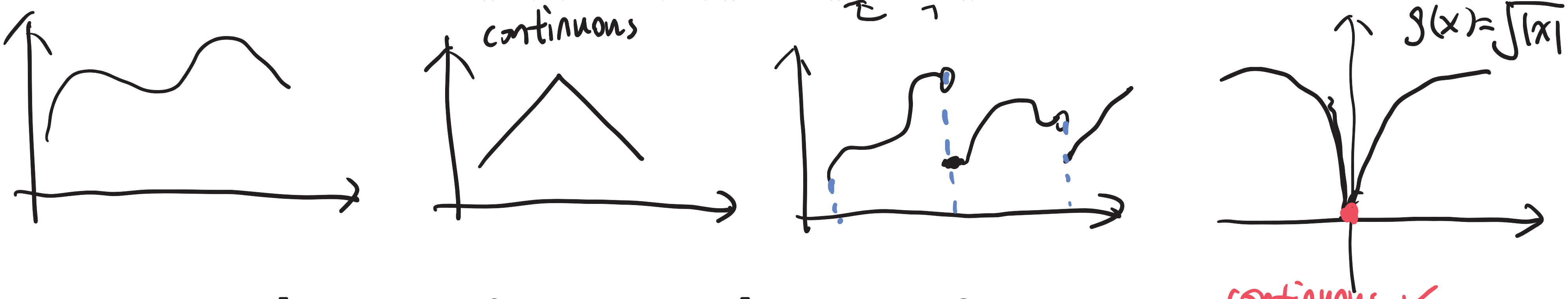
  - $x \in \mathbb{R}^n$: system state, $u \in \mathbb{R}^m$: control input, $y \in \mathbb{R}^p$: system output
  - A, B, C, D are constant matrices with appropriate dimensions

Existence and Uniqueness of ODE Solutions

- Function $g : \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* over domain $\mathcal{D} \subseteq \mathbb{R}^n$ if $\exists L < \infty$

"smooth"

$$\|g(x) - g(x')\| \leq L\|x - x'\|, \forall x, x' \in \mathcal{D}$$



continuous ✓

NOT Lipschitz.

- **Theorem [Existence & Uniqueness]** Nonlinear ODE

$$① : \quad \dot{x}(t) = f(x(t), t), \quad \text{I.C. } x(t_0) = x_0$$

has a *unique* solution if $f(x, t)$ is Lipschitz in x and piecewise continuous in t

$$\Downarrow$$

$$\|f(x,t) - f(x',t)\| \leq L \cdot \|x - x'\|, \quad \forall t \in [t_0, t_f] \quad \text{(for all)}$$

· solution to ① : means: function of t, $\hat{x}(t)$

$$\Leftrightarrow \begin{cases} \hat{x}(t_0) = x_0 \\ \dot{\hat{x}}(t) = f(\hat{x}(t), t), \ \forall t \end{cases}$$

Existence and Uniqueness of Linear Systems

- **Corollary**: Linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

has a unique solution for any piecewise continuous input $u(t)$

· vector field func: $\underline{f(x,t)} = \underline{Ax + Bu(t)}$

proof: check condition: ① $\|f(x,t) - f(x',t)\| = \|A(x-x')\| \le \|A\| \|x-x'\|$

L ✓

② $f(x,t) = Ax + Bu(t)$ is piecewise continuous in t?

Yes, because $u(t)$ is piecewise continuous in t (as stated in the condition)

- *Homework*: Suppose A becomes time-varying $A(t)$, can you derive conditions to ensure existence and uniqueness of $\dot{x}(t) = A(t)x(t) + Bu(t)$?

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How to Solve Linear Differential Equations?

$x(t) \in \mathbb{R}^n$ $\quad d(t) = B u(t)$

- General linear ODE: $\dot{x}(t) = Ax(t) + d(t)$
- The key is to derive solutions to the autonomous linear case: $\dot{x}(t) = Ax(t)$, with $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and initial condition (IC) $x(0) = x_0$.
- By existence and uniqueness theorem, the ODE $\dot{x} = Ax$ admits a unique solution.
- It turns out that the solution can be found analytically via the *Matrix Exponential*

What is the "Euler's Number" e?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t) \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution: $z(t) = e^{at} \cdot z_0$

proof: check I.C. $z(0) = z_0$ ✓

check vector field: $\dot{z}(t) = (e^{at} z_0)' = a(e^{at} z_0) = a \cdot z(t)$ ✓

- What is the number "e"?

- Euler's number

- Defined as the number such that $\boxed{(e^x)' = e^x}$, $(2^x)' \neq 2^x$, $(3^x)' \neq 3^x$

$$\Rightarrow \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \Rightarrow \frac{e^h - 1}{h} \xrightarrow{h\to 0} 1$$

$$\Rightarrow e^h \to h+1 \Rightarrow e = \lim_{h\to 0} (h+1)^{1/h}$$

$\approx 2.71\ldots$

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}$$

$$(1 + x + \frac{x^2}{2!} + \cdots)' = (0 + 1 + x + \frac{x^2}{2!} + \cdots)$$

- This can be extended to complex variables:

$$e^z \triangleq \sum_{k=0}^{\infty} \frac{z^k}{k!} = \underline{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

let $z = j\theta$

This power series is well defined for all $z \in \mathbb{C}$

- In particular, we have $e^{j\theta} = \underline{1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots}$ $= \cos\theta + j\sin\theta$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

$\sin\theta$ Taylor expansion $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

$\cos\theta$ . . . . $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

$e^{j\theta} = \cos\theta + j\sin\theta$

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential* for square $A \in \mathbb{R}^{n \times n}$

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = \underbrace{I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots}_{} \in \mathbb{R}^{n \times n}$$

e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdots\cdots$

$$e^A = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{bmatrix} + \cdots = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$

- This power series is well defined for any finite square matrix $A \in \mathbb{R}^{n \times n}$.

Some Important Properties of Matrix Exponential

By definition

① • $Ae^A = e^A A$ proof:

$$A \sum_{i=0}^{\infty} \frac{A^i}{i!} = \sum \frac{A^i}{i!} A$$

$Ae^B \neq e^B A$, if $AB \neq BA$

② • $\boxed{e^A e^B = e^{A+B} \text{ if } AB = BA}$

③ • If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$

P nonsingular $A \sim D$.

$$e^A = I + PDP^{-1} + \frac{PDP^{-1}PDP^{-1}}{2!} + \cdots$$

PIP^{-1}

PD^2P^{-1}

$$= Pe^D P^{-1}$$

④ • For every $t, \tau \in \mathbb{R}$, $e^{At}e^{A\tau} = e^{A(t+\tau)}$

immediate from ②

$$\underbrace{(At)}_{A_1}\underbrace{(A\tau)}_{A_2} = A_2 A_1$$

⑤ • $(e^A)^{-1} = e^{-A}$

From 2, $\underbrace{e^A}_{T} \underbrace{e^{-A}}_{P} = e^{A+(-A)} = e^{[0]} = I$

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$$
\begin{array}{ccc}
e^{x} & \longrightarrow & e^{A} \\
\Downarrow & & \downarrow \\
\dot{x} = ax & & \dot{x} = Ax
\end{array}
$$

$x \in \mathbb{R}^n$

Autonomous Linear Systems

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.
- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$\boxed{x(t) = e^{At} x_0}$$

function t

$\in \mathbb{R}^n$

Proof: check I.C.: $x(0) = e^{A \cdot 0} x_0 = x_0$ ✓

check vector field: $\frac{d}{dt}(e^{At} x_0) \stackrel{?}{=} A e^{At} x_0$

By definition: $\frac{d}{dt}(e^{At} x_0) = \frac{d}{dt}\left(I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\right) x_0$

$= \left(A + A^2 t + A^3 \frac{t^2}{2!} + \cdots\right) x_0$

$\| $

Computation of Matrix Exponential (1/2)

- Directly from definition : $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

$$= A\underbrace{\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)x_0}$$

$$= A(e^{At}x_0)$$

· It's hard to compute by hand

· For special case, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ → nilpotent , $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 = [0]$

$$e^A = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- For diagonalizable matrix:

Example: $A = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$

$$\Rightarrow At = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{P^{-1}} \xRightarrow{\text{prop } 3} e^{At} = \begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{2t}, & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Computation of Matrix Exponential (2/2)

scalar:
$$\hat{X}(s) = \int x(t) e^{-st} dt$$

- Using Laplace transform

Laplace transform

- $\dot{x} = Ax$, $x(0) = x_0 \in \mathbb{R}^n$

$\mathcal{L}(x(t))$: $x(t) \longleftrightarrow \hat{X}(s) \in \mathbb{C}^n$

$\in \mathbb{R}^n$

$\dot{x}(t) \longleftrightarrow s\hat{X}(s) - x_0$

Apply L.T. $\Rightarrow$ $s\hat{X}(s) - x_0 = A\hat{X}(s)$

$$(sI - A)\hat{X}(s) = x_0$$

$$\Rightarrow \hat{X}(s) = (sI - A)^{-1} x_0$$

$$\Rightarrow X(t) = \mathcal{L}^{-1}\left((sI - A)^{-1} x_0\right)$$

Solution to General Linear Systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3}$$

- $x \in \mathbb{R}^n$ is system state, $u \in \mathbb{R}^m$ is control input, $y \in \mathbb{R}^p$ is the system output

- A, B, C, D are constant matrices with appropriate dimensions

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$

- **Homework:** The solution to the linear system (3) is given by

I. C.

vector

$$\begin{cases} x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{cases}$$

More Discussions