MEE5114 Advanced Control for Robotics

Lecture 7: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

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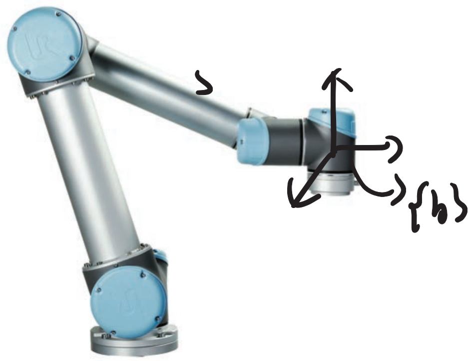
Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics

to characterize {b}'s pose.

FK: Find the func : $T_b(\theta_1, \theta_2, \cdots, \theta_n) \in SE(3)$



Result: $T_b(\theta_1, \cdots, \theta_n) = e^{[{}^0S_1]\theta_1} e^{[{}^0S_2]\theta_2} \cdots e^{[{}^0S_n]\theta_n} M$

means: screw axis of joint 2 expressed in {s} frame

when at home position

- **Velocity Kinematics**: How does the velocity of {b} relate to the joint velocities $\dot{\theta}_1, \ldots, \dot{\theta}_n$; {b}'s velocity is due to joints' velocities.
- This depends on how to represent {b}'s velocity : Can we use $\dot{T}_b$ to represent velocity of {b} ? (4×4)
  - Twist representation → **Geometric Jacobian**

$\mathcal{V}_b = \begin{bmatrix} \omega \\ v \end{bmatrix}$, $\mathcal{V}_b(\theta, \dot{\theta})$, it turns out, $\mathcal{V}_b$ is a linear func of $\dot{\theta} \Rightarrow \mathcal{V}_b = J(\theta)\dot{\theta}$

$6\times n$ matrix

this matrix is the Geometric Jacobian

  - Local coordinate of SE(3) → **Analytic Jacobian**

$$\theta_1, \cdots, \theta_n \xmapsto{\text{F.K.}} \underbrace{T_b(\theta_1, \cdots, \theta_n)}_{\in SE(3)} = (R, p) \xmapsto{\text{local coordinate}} x = \begin{bmatrix} p_x(\theta) \\ p_y(\theta) \\ p_z(\theta) \\ \alpha(\theta) \\ \beta(\theta) \\ \gamma(\theta) \end{bmatrix} \in \mathbb{R}^6$$

p: x, y, z

R: α, β, γ

rpy

e.g.

all depend on $\theta_1 \cdots \theta_n$

$g(\cdot)$

Outline: choose "task variable": (quantities of interest)

e.g. $x = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \in \mathbb{R}^3$ or $x = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \to$ orientation of {b}

only cares about origin of {b}

$x = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$

$x = p_y$

In all cases, we can derive

$$\boxed{x = g(\theta_1, \cdots, \theta_n)}$$

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$x \in \mathbb{R}^{n_x}$

$\theta \in \mathbb{R}^n$

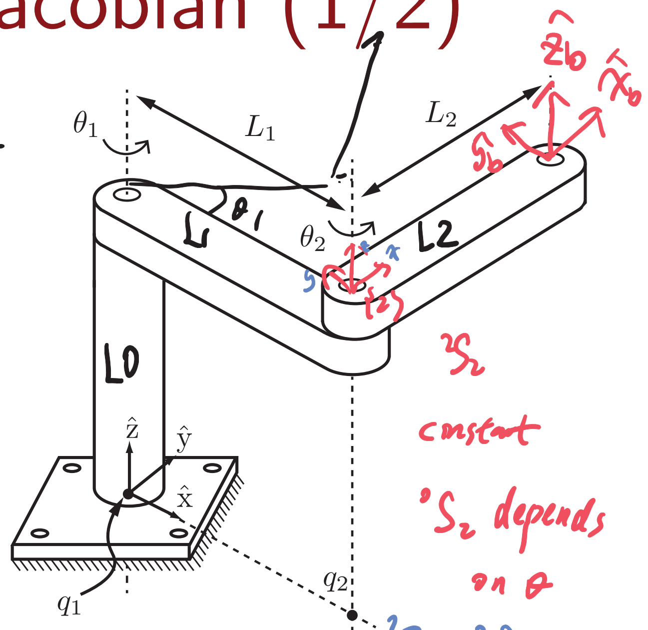
$$\Rightarrow \dot{x} = \frac{d}{dt}\left(g(\theta_1, \cdots, \theta_n)\right)$$

$$= \frac{\partial g}{\partial \theta} \cdot \dot{\theta}$$

$\underbrace{n_x \times n \text{ matrix}}$

Analytic Jacobian

Simple Illustration Example: Geometric Jacobian (1/2)



$^{2}S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- Coordinate free: Joint 1 Joint 2

· two joint axis: screw axis 1: $\underbrace{S_1}_{\text{constant}}$ $S_2(\theta_1)$

- Goal: find twist of {b}, given $\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2$

Link 0: $\mathcal{V}_{L0} = 0 \in \mathbb{R}^6$; Link 1: $\mathcal{V}_{L1} = S_1 \dot{\theta}_1$

Link 0: $\mathcal{V}_{L2} = \mathcal{V}_{L2/L1} + \mathcal{V}_{L1/L0} = S_2\dot{\theta}_2 + S_1\dot{\theta}_1 = \begin{bmatrix} S_1 & \underbrace{S_2(\theta)} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

screw axis of Joint 2

{b}: $\mathcal{V}_b = \mathcal{V}_{L2} = \underbrace{\begin{bmatrix} S_1 & \vdots & S_2(\theta) \end{bmatrix}}_{\text{Geometric Jacobian}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} J_1(\theta) & \vdots & J_2(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

1st-column of Geo-Jacobian : screw axis of Joint 1

Simple Illustration Example: Geometric Jacobian (2/2)

- (Side Note: in General $\mathcal{V}_b = [J_1(\theta) \cdots J_n(\theta)] \begin{bmatrix} \dot\theta_1 \\ \vdots \\ \dot\theta_n \end{bmatrix}$

$= J_1(\theta)\cdot\dot\theta_1 + J_2(\theta)\,\dot\theta_2 + \cdots + J_n(\theta)\,\dot\theta_n$

$J_2(\theta)\,\dot\theta_2$ = "only θ_2 changes, all other joints do not move." the velocity of {b} when)

- Computation: Let's work with {o}
- let ${}^oS_1(\theta) = {}^oS_1(\theta=0) \triangleq {}^o\bar S_1 = \begin{bmatrix} 0\\0\\1\\0\\0\\0 \end{bmatrix}$
- let $\theta_1 = 0$, ${}^o\bar S_2 = \begin{bmatrix} 0\\0\\1\\0\\-L_1\\0 \end{bmatrix}$

$${}^oV_2 = -{}^o\omega \times {}^oq_2 = \begin{bmatrix} 0\\0\\-1 \end{bmatrix} \times \begin{bmatrix} L_1\\0\\0 \end{bmatrix} = \begin{bmatrix} 0\\-L_1\\0 \end{bmatrix}$$

- when $\theta_1 \neq 0$, we need ${}^oS_2(\theta)$

$${}^o\bar S_2 = {}^oS_2(\theta_1=0) \xmapsto{\hat T_1 = e^{[{}^o\bar S_1]\theta_1}} {}^oS_2(\theta) = \underbrace{[Ad_{\hat T(\theta_1)}]}_{6\times 6}\,\underbrace{{}^o\bar S_2}_{6\times 1}$$

$$\Rightarrow {}^oJ(\theta) = [\,{}^o\bar S_1 \;\vdots\; [Ad_{\hat T(\theta_1)}]\,{}^o\bar S_2\,]$$

$\leftarrow$ all we need for computation is ${}^o\bar S_1$ ${}^o\bar S_2$

Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that we have n joints

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n = \begin{bmatrix} J_1(\theta) & J_2(\theta) & \cdots & J_n(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$J_i(\theta)$: screw axis of joint i when robot at θ config.

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

Geometric Jacobian: General Case (2/3)

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.

- The simplest way to write Jacobian is to use local coordinate:

$$^iJ_i = {}^iS_i, \quad i = 1, \ldots, n$$

- In fixed frame $\{0\}$, we have

$$^0J_i(\theta) = {}^0X_i(\theta)\, {}^iS_i, \quad i = 1, \ldots, n \tag{1}$$

- Recall: ^0X_i is the change of coordinate matrix for spatial velocities.

$^0T_i(\theta=0) \neq {}^0T_b(\theta=0)$

- Assume $\theta = (\theta_1, \ldots, \theta_n)$, then

$$^0T_i(\theta) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^0X_i(\theta) = \left[\mathrm{Ad}_{{}^0T_i(\theta)}\right] \tag{2}$$

pose of {i}'s relative 0

Geometric Jacobian: General Case (3/3)

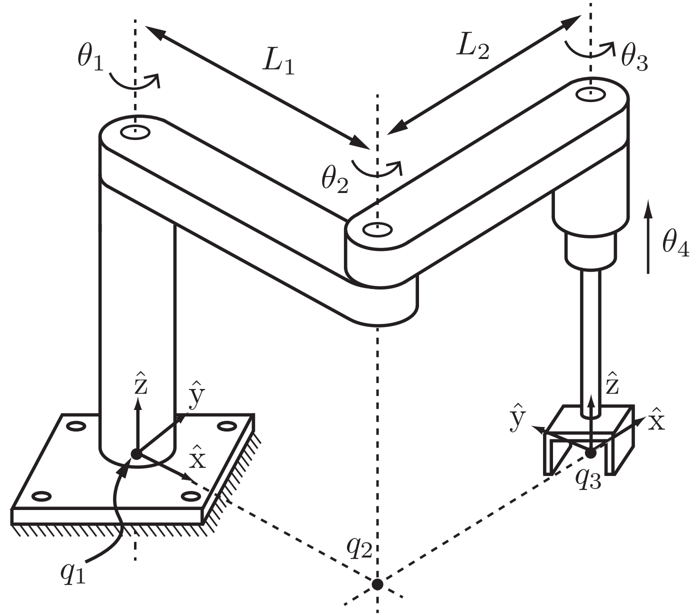
- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula

  ↳ only needs ${}^0\bar{S}_i$

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i=1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i=2$, ${}^0S_2(\theta) = {}^0S_1(\theta_1) = \left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$
  - For general i, we have

$$
\begin{aligned}
{}^0J_i(\theta) = {}^0S_i(\theta) &= \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i \\
\text{where} \quad \hat{T}(\theta_1,\ldots,\theta_{i-1}) &\triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}}
\end{aligned} \tag{3}
$$

Geometric Jacobian Example



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Analytic Jacobian

${}^0T_b = ({}^0R_b, {}^0p_b)$ Cartesian / Spherical coordinate

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be Cartesian + Euler angle of end-effector frame
    (x, y) or (ρ, θ) ZYZ ZYX ...
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame
    $x = g(\theta)$ joint variable $\dot{x} = \left(\frac{\partial g}{\partial \theta}\right)\dot{\theta}$
- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = E(\theta)J(\theta)$$

  - $E(x)$ can be easily found with given parameterization x

Simple Illustration Example: Analytic Jacobian (1/3)

choose task space variable ${}^0p_b$

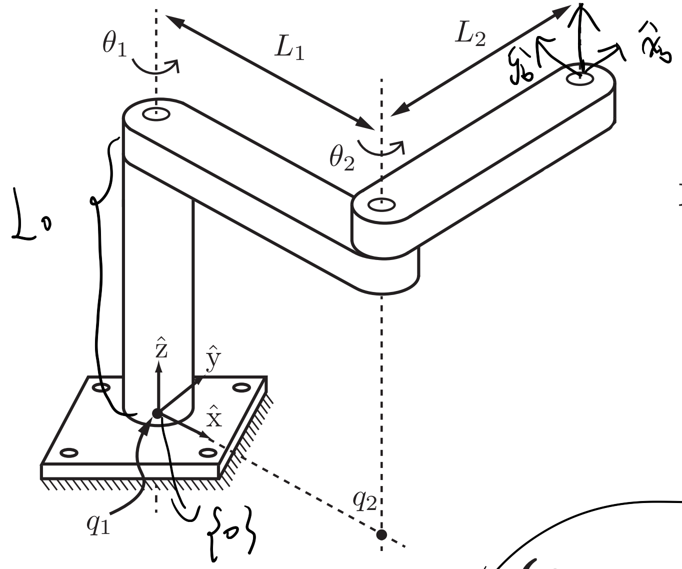
${}^0p_b = \underline{g}(\theta)$

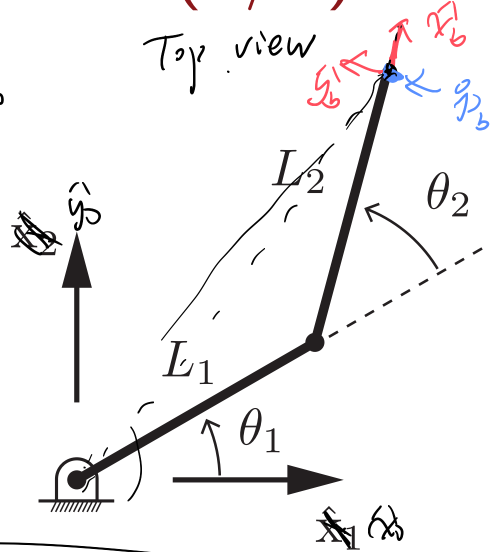
Note: if we use polar coordinate

$${}^0p_b = \begin{bmatrix} {}^0\rho_b \\ {}^0\phi_b \end{bmatrix} = \hat{g}(\theta)$$

$${}^0\dot{p}_b = \underbrace{\left(\frac{\partial g}{\partial \theta}\right)}_{J_a(\theta)} \dot{\theta}$$

$J_a(\theta)$: analytical Jacobian.





$$\begin{cases} {}^0p_{b,x} = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) \\ {}^0p_{b,y} = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) \\ {}^0p_{b,z} = L_0 \quad \cdots \; g_3 \end{cases}$$

$$J_a(\theta) = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \\ \frac{\partial g_3}{\partial \theta_1} & \frac{\partial g_3}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix}$$

3x2 ($n_x \times n$

dim of task space variable

joint numbers

Simple Illustration Example: Analytic Jacobian (2/3)

- Let ${}^0J_b(\theta)$ ~~dno~~ denote the geometric Jacobian of $\{b\}$

$${}^0\mathcal{V}_b = {}^0J_b(\theta)\,\dot\theta \quad , \qquad {}^0\mathcal{V}_b = \begin{bmatrix} {}^0\omega \\ {}^0v \end{bmatrix}$$

$${}^0\dot p_b = {}^0v + {}^0\omega \times {}^0p_b = -{}^0p_b \times {}^0\omega + {}^0v = \underbrace{\begin{bmatrix} -[{}^0p_b] & I_{3\times3} \end{bmatrix}}\underbrace{\begin{bmatrix} {}^0\omega \\ {}^0v \end{bmatrix}}$$

$$= \begin{bmatrix} -[{}^0p_b] & I_{3\times3} \end{bmatrix} {}^0J_b(\theta)\,\dot\theta$$

J_a

Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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