MEE5114 Advanced Control for Robotics

Lecture 8: Rigid Body Dynamics

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Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Acceleration

- Given a rigid body with spatial velocity $\mathcal{V} = (\omega, v_o)$, its spatial acceleration is

coordinate free

$$\mathcal{A} = \dot{\mathcal{V}} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix} \qquad A \triangleq \lim_{\delta \to 0} \frac{\mathcal{V}(t+\delta) - \mathcal{V}(t)}{\delta}$$

  - Recall that: v_o is the velocity of the body-fixed particle coincident with frame origin o at the current time t.

    $q \to$ body-fixed $\quad v_o = \dot{q}$

  - Note: $\dot{\omega}$ is the angular acceleration of the body

at time t: $o = q(t) \quad v_o = \dot{q}(t)$, but $\dot{v}_o \neq \ddot{q}(t)$

  - $\dot{v}_o$ is not the acceleration of any body-fixed point!

  - In fact, $\dot{v}_o$ gives the rate of change in stream velocity of body-fixed particles passing through o

Spatial vs. Conventional Accel. (1/2)

- Why "$\dot{v}_o$ is not the acceleration of any body-fixed point"?

- Suppose $q(t)$ is the body fixed particle coincides with o at time t_0

- So by definition, we have $v_o(t_0) = \dot{q}(t_0)$, however, $\dot{v}_o(t_0) \neq \ddot{q}(t_0)$, where $\ddot{q}(t_0)$ is the conventional acceleration of the body-fixed point q

  - Note: $\dot{v}_o(t_0) \triangleq \lim_{\delta \to 0} \frac{v_o(t_0+\delta) - \overbrace{v_o(t_0)}^{=\dot{q}(t_0)}}{\delta}$

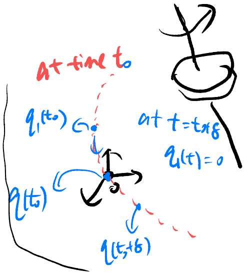
At time $t = t_0$, $q(t_0) = $ "o" $\Rightarrow V_o(t_0) = \dot{q}(t_0)$

At time $t = t_0+\delta$, $q(t) \neq$ "o" $\Rightarrow V_o(t_0+\delta) \neq \dot{q}(t_0+\delta)$

Assume: at $t = t_0+\delta$, $q_1(t_0+\delta) = o$, $\Rightarrow V_o(t_0+\delta) = \dot{q}_1(t_0+\delta)$

Note: q_1 and q are different points:

$$\dot{q}_1(t_0+\delta) \neq \dot{q}(t_0+\delta)$$



Spatial vs. Conventional Accel. (2/2)

should be $\dot{q}_1(t_0+\delta)$

so $\dot{v}_o(t_0) = \lim_{\delta \to 0} \frac{v_o(t_0+\delta) - v_o(t_0)}{\delta} \neq \lim_{\delta \to 0} \frac{\dot{q}(t_0+\delta) - \dot{q}(t_0)}{\delta} = \ddot{q}(t_0)$

By definition: at all t, we have $\dot{q}(t) = v_o(t) + \omega(t) \times \vec{oq}(t)$

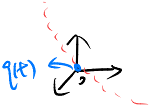
we also know (assumed), $q(t_0) = o \Rightarrow \vec{oq}(t_0) = 0$

$$\Rightarrow \boxed{\ddot{q}(t) = \dot{v}_o(t) + \dot{\omega}(t) \times \vec{oq}(t) + \omega(t) \times \dot{q}(t)}$$

At $t = t_0$; $\vec{oq} = 0 \Rightarrow \ddot{q}(t_0) = \dot{v}_o(t_0) + \omega(t_0) \times \dot{q}(t_0)$

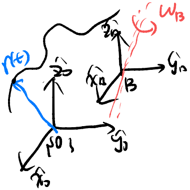
- If $q(t)$ is the body fixed particle coincides with o at time t, then we have

$$\boxed{\ddot{q}(t) = \dot{v}_o(t) + \omega(t) \times \dot{q}(t)}$$



Plücker Coordinate System and Basis Vectors (1/3)

- Recall coordinate-free concept: let $r \in \mathbb{R}^3$ be a free vector with {o} and {B} frame coordinate ${}^o r$ and ${}^B r$



$${}^o r = \begin{bmatrix} {}^o r_x \\ {}^o r_y \\ {}^o r_z \end{bmatrix} \in \mathbb{R}^3 \Longleftrightarrow r = [\hat{x}_o \ \ \hat{y}_o \ \ \hat{z}_o]\, {}^o r \quad ①$$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ number

$${}^B r = \begin{bmatrix} {}^B r_x \\ {}^B r_y \\ {}^B r_z \end{bmatrix} \in \mathbb{R}^3 \Longleftrightarrow r = [\hat{x}_B \ \ \hat{y}_B \ \ \hat{z}_B]\, {}^B r \quad ②$$

$\Downarrow$

express this "physics" in "o" frame

$${}^o r = \underbrace{[{}^o\hat{x}_B \ \ {}^o\hat{y}_B \ \ {}^o\hat{z}_B]}_{{}^oR_B}\, {}^B r$$

$$① \Rightarrow \dot{r} = [\hat{x}_o \ \ \hat{y}_o \ \ \hat{z}_o]\, \frac{d}{dt}({}^o r)$$

apparent derivative

$\triangleq {}^o\dot{r}$

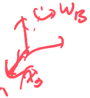
use {o}-frame to express physics:

$${}^o(\dot{r}) = [{}^o\hat{x}_o \ \ {}^o\hat{y}_o \ \ {}^o\hat{z}_o]\, \frac{d}{dt}({}^o r) = \frac{d}{dt}({}^o r)$$

$$\boxed{{}^o(\dot{r}) = {}^o\dot{r}}$$

$${}^o\left[\frac{d}{dt}(r)\right] \qquad \frac{d}{dt}({}^o r)$$

Plücker Coordinate System and Basis Vectors (2/3)



- ② $\Rightarrow$ $r = [\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]\,{}^B r$ - -

  ${}^B(\dot{r}) \overset{?}{=} \underbrace{({}^B r)'}_{{}^B\mathring{r}}$ $\times$ if $\{B\}$ is changing

$\dot{\hat{x}}_B = \omega_B \times \hat{x}_B$

${}^D R_B$

$$\boxed{\begin{aligned}\dot{r} &= [\dot{\hat{x}}_B \;\; \dot{\hat{y}}_B \;\; \dot{\hat{z}}_B]\,{}^B r + [\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]({}^B r)' \\ &= \omega_B \times [\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]\,{}^B r + [\hat{x}_B \;\; \hat{y}_B \;\; \hat{z}_B]({}^B r)'\end{aligned}}$$

use $\{B\}$ frame to express "physics"

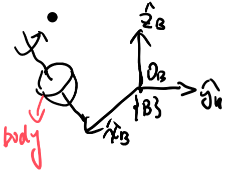
$${}^B(\dot{r}) = \underbrace{{}^B\omega_B \times {}^B r}_{\text{accounts for coordinate frame is moving}} + \frac{d}{dt}({}^B r) \triangleq {}^B\mathring{r}$$

due to changes in coordinate

${}^B r = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $r = \hat{x}_B$

$= \begin{bmatrix}0\\1\\1\end{bmatrix}$, $r = \hat{y}_B + \hat{z}_B$

Plücker Coordinate System and Basis Vectors (3/3)



Given $\{B\}$ frame

$\{e_{B_1}, e_{B_2} \cdots e_{B_6}\}$ -

6 - motion basis vectors

each twist is linear combination of 6 - motion basis vectors

$${}^B\mathcal{V}_{body} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$ means "body" rotates about $\hat{x}_B$ at unit speed.

$\Longleftrightarrow \mathcal{V}_{body} \triangleq e_{B_1} \leftarrow$ motion basis vector

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$ means "body" rotates about $\hat{y}_B$ at unit speed

$\Longleftrightarrow \mathcal{V}_{body} \triangleq e_{B_2}$

$\vdots$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$ means "body" linearly moves along $\hat{x}_B$ at unit speed

$\Longleftrightarrow \mathcal{V}_{body} \triangleq e_{B_4}$

$$\underbrace{\mathcal{V}_{body}}_{\text{coordinate free.}} = \alpha_1 e_{B_1} + \alpha_2 e_{B_2} + \cdots + \alpha_6 e_{B_6}, \quad \text{where} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \end{bmatrix} = {}^B\mathcal{V}_{body}$$

Work with Moving Reference Frame

- Use $\{o\}$-frame to express the above physics

$$ {}^{o}\mathcal{V}_{body} = \alpha_1 \, {}^{o}e_{B_1} + \alpha_2 \, {}^{o}e_{B_2} + \cdots + \alpha_6 \, {}^{o}e_{B_6} = \begin{bmatrix} {}^{o}e_{B_1} & {}^{o}e_{B_2} & \cdots & {}^{o}e_{B_6} \end{bmatrix} {}^{B}\mathcal{V}_{body} $$

${}^{o}e_{B_i}$: can be computed from "physics" and twist definition

e.g. ${}^{o}e_{B_1}$: unit speed rotation about $\hat{x}_B$ axis expressed in $\{o\}$

- $\underbrace{\begin{bmatrix} {}^{o}e_{B_1} & \cdots & {}^{o}e_{B_6} \end{bmatrix}}_{6\times 6} = {}^{o}X_B = [Ad_{T_B}]$

If ${}^{o}T_B = (R, p)$

$$ {}^{o}X_B = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} $$

- $A_{body} \triangleq \frac{d}{dt}(\mathcal{V}_{body})$ $\quad$ $\mathcal{V}_{body} = [e_{B_1}\ e_{B_2} \cdots e_{B_6}]\ {}^B\mathcal{V}_{body}$

$$\Rightarrow A_{body} = [\dot{e}_{B_1}\ \dot{e}_{B_2} \cdots \dot{e}_{B_6}]\ {}^B\mathcal{V}_{body} + [e_{B_1} \cdots e_{B_6}]\ \frac{d}{dt}({}^B\mathcal{V}_{body})$$

$\triangleq {}^B\dot{\mathcal{V}}^{\circ}_{body}$

① • If $\{B\}$ does not change (e.g. $\{0\}$ frame case)

$$\dot{\mathcal{V}}_{body} = [e_{B_1} \cdots e_{B_6}]\ {}^B\dot{\mathcal{V}}^{\circ}_{body} \xRightarrow{\text{express in } \{B\}} {}^B\dot{\mathcal{V}}_{body} = {}^B\dot{\mathcal{V}}^{\circ}_{body}$$

② If $\{B\}$ changes over time, then $\{B\}$ has twist $\mathcal{V}_B$

${}^B\left(\frac{d}{dt}(\mathcal{V}_{body})\right)$

- ${}^BA_{body} \neq {}^B\dot{\mathcal{V}}^{\circ}_{body}$

$= {}^BA$

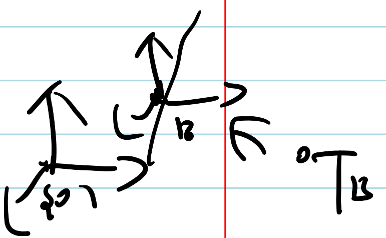
The key is to compute $[\dot{e}_{B_1}\ \dot{e}_{B_2} \cdots \dot{e}_{B_6}] \leftarrow$ can be computed purely by physics (see Featherstone)

- Now let's work with "0" frame to find

$$\Rightarrow \text{we need to compute } [{}^0\dot{e}_{B_1}\ {}^0\dot{e}_{B_2} \cdots {}^0\dot{e}_{B_6}] = {}^0\dot{X}_B = \frac{d}{dt}\left[Ad_{T_B}\right]$$

$R = [x_B \; y_n \; z$

Let's denote ${}^0T_B = [R, p] \Rightarrow {}^0\dot{X}_B = \frac{d}{dt}\left(\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}\right)$



$\{B\}$- frame has instantaneous velocity $\mathcal{V}_B = \begin{bmatrix} \omega \\ v_0 \end{bmatrix}$

$$\dot{R} = \omega \times R, \quad ([p]R)' = \underbrace{[\dot{p}]R + [p]\dot{R}}$$

Note: $\dot{p} = v_0 + \omega \times \vec{op}$

$$[\dot{p}] = [v_0 + \omega \times p]$$

$$= [v_0] + \underbrace{[\omega \times p]}$$ ← Jacobi's identity

$$= [v_0] + [\omega][p] - [p][\omega]$$

$$= [v_0]R + [\omega][p]R - \cancel{[p][\omega]R} + \cancel{[p][\omega]}R$$

$$= [v_0]R + [\omega][p]R$$

$$\Rightarrow \boxed{{}^0\dot{X}_B} = \begin{bmatrix} [\omega]R & 0 \\ [v_0]R + [\omega][p]R & [\omega]R \end{bmatrix} = \begin{bmatrix} [\omega] & 0 \\ [v_0] & [\omega] \end{bmatrix} {}^0X_B$$

$$\left({}^{0}\dot{X}_B\right) = \underbrace{\begin{bmatrix} [w] & 0 \\ [v_0] & [w] \end{bmatrix}}\ {}^{0}X_B \triangleq {}^{0}\mathcal{V}_B \times {}^{0}X_B$$

6-dim / twist cross / spatial cross product.

$${}^{0}\dot{R}_B = {}^{0}\omega_B \times {}^{0}R_B = [{}^{0}\omega_B]\ {}^{0}R_B$$

Notation: Given $\mathcal{V}_1 = \begin{bmatrix} w \\ v \end{bmatrix}$, $[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [w] & 0 \\ [v] & [w] \end{bmatrix}$

Recall: $[\mathcal{V}] = \begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix} \in se(3)$

In coordinate free: $R_B = [\hat{x}_B \ \ \hat{y}_B \ \ \hat{z}_B]$, $\{B\}$: ω_B rotation

$$\boxed{\dot{R}_B = \omega_B \times R_B}$$

$X_B = [e_{B_1} \ \ e_{B_2} \cdots e_{B_6}]$, $\{B\}$ has twist $\mathcal{V}_B$

$$\boxed{\dot{X}_B = \mathcal{V}_B \times X_B}$$

$$\dot{e}_{B_1} = \mathcal{V}_B \times e_{B_1}$$

Derivative of Adjoint

- Suppose a frame {A}'s pose is $T_A = (R_A, p_A)$, and is moving at an instantaneous velocity $\mathcal{V}_A = (\omega, v)$. Then

$$\underbrace{\frac{d}{dt}\left([\mathrm{Ad}_{T_A}]\right)} = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} [\mathrm{Ad}_{T_A}]$$

$$\boxed{\underbrace{\frac{d}{dt}(X_A) = \mathcal{V}_A} \times X_A}$$

$${}^{0}\dot{X}_A = {}^{0}\mathcal{V}_A \times {}^{0}X_A$$

$${}^{A}\left(\dot{X}_A\right) = {}^{A}\mathcal{V}_A \times {}^{A}X_A$$

Spatial Cross Product

- Given two spatial velocities (twists) $\mathcal{V}_1$ and $\mathcal{V}_2$, their spatial cross product is:

$$\underbrace{\mathcal{V}_1 \times \mathcal{V}_2}_{\text{Lie Bracket}} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \times \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}$$

- Matrix representation: $\mathcal{V}_1 \times \mathcal{V}_2 = [\mathcal{V}_1 \times]\mathcal{V}_2$, where

$$[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}$$

- Roughly speaking, when a motion vector $\mathcal{V}$ is moving with a spatial velocity $\mathcal{Z}$ (e.g. it is attached to a moving frame) but is otherwise not changing, then

$$\dot{\mathcal{V}} = \mathcal{Z} \times \mathcal{V}$$

Spatial Cross Product: Properties (1/1)

- Assume A is moving wrt to O with velocity $\mathcal{V}_A$

$$^O\dot{X}_A = [^O\mathcal{V}_A \times]^O X_A$$

another twist $\mathcal{V}_1$

- $[X\mathcal{V}\times] = X[\mathcal{V}\times]X^T$, for any transformation X and twist $\mathcal{V}$

$\Updownarrow$

$[R \cdot w] = R[w]R^T$

Spatial Acceleration with Moving Reference Frame

Consider a body with velocity $\mathcal{V}_{body}$ (wrt inertia frame), and ${}^{O}\mathcal{V}_{body}$ and ${}^{B}\mathcal{V}_{body}$ be its Plücker coordinates wrt {O} and {B}:

- $${}^{B}\mathcal{A}_{body} = \frac{d}{dt}\left({}^{B}\mathcal{V}_{body}\right) + \underbrace{{}^{B}\mathcal{V}_{B} \times {}^{B}\mathcal{V}_{body}}_{\text{due to frame \{B\} is moving}}$$

$\Downarrow$ ${}^{B}\left(\frac{d}{dt}(\mathcal{V}_{body})\right)$ $\qquad \Downarrow$ apparent derivative ${}^{B}\mathring{\mathcal{V}}_{body}$

$\mathcal{V}_B$: frame {B} twist

$\mathcal{V}_{body}$: body velocity twist

- $${}^{O}\mathcal{A} = {}^{O}X_B\, {}^{B}\mathcal{A} \rightarrow {}^{B}\left(\frac{d}{dt}(\mathcal{V}_{body})\right)$$

${}^{O}\left(\frac{d}{dt}(\mathcal{V}_{body})\right)$

{O} frame does not change

$${}^{O}(A_{body}) = \frac{d}{dt}({}^{O}\mathcal{V}_{body}) = \frac{d}{dt}({}^{O}X_B\, {}^{B}\mathcal{V}_{body}) = {}^{O}\dot{X}_B\, {}^{B}\mathcal{V}_{body} + {}^{O}X_B\, {}^{B}\mathring{\mathcal{V}}_{body}$$

$$= [{}^{O}\mathcal{V}_B \times]\, {}^{O}X_B\, {}^{B}\mathcal{V}_{body} + {}^{O}X_B\, {}^{B}\mathring{\mathcal{V}}_{body}$$

$$= {}^{O}X_B \left\{ {}^{B}X_O [{}^{O}\mathcal{V}_B \times]\, {}^{O}X_B\, {}^{B}\mathcal{V}_{body} + {}^{B}\mathring{\mathcal{V}}_{body} \right\} \cdots (*)$$

Spatial Acceleration Example

- Find ${}^B A_{top} = {}^B\left(\frac{d}{dt}\mathcal{V}_{top}\right)$

Method 1: $= {}^B\dot{\mathcal{V}}_{top} + {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{top}$

$${}^B\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} \qquad {}^B\omega = \begin{bmatrix} 0 \\ 0 \\ 50 \text{ rad/s} \end{bmatrix} \qquad {}^B v_B = \begin{bmatrix} 0 \\ 0.02 \\ 0 \end{bmatrix}$$

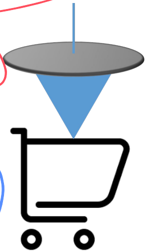
$${}^B A_{top} = \frac{d}{dt}\left({}^B\mathcal{V}_{top}\right) + {}^B\mathcal{V}_{Bframe} \times {}^B\mathcal{V}_{top}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

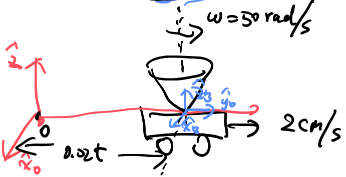
$$\left[{}^B X_O\, {}^O\mathcal{V}_B \times\right] = \left[{}^B\mathcal{V}_B \times\right]$$

$$\Rightarrow \quad (*) = {}^O X_B \left\{ {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body} + {}^B\dot{\mathcal{V}}_{body} \right\}$$

$${}^B\left(\frac{d}{dt}\left(\mathcal{V}_{body}\right)\right)$$



${}^B\mathcal{V}_{body}$



$$\begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}$$

Outline

Method 2: use {o} frame

$$ {}^{o}A = \frac{d}{dt}\left({}^{o}\mathcal{V}_{top}\right), \quad {}^{o}\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 5\pi \\ \hline t \\ 0.02 \\ 0 \end{bmatrix} \qquad v_0 = \begin{bmatrix} 50\times 0.02t \\ 0.02 \\ 0 \end{bmatrix} $$

- Spatial Acceleration
- **Spatial Force (Wrench)**
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

$$ {}^{o}A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{B}A = {}^{B}X_{o} \cdot {}^{o}A $$

$$ = \begin{bmatrix} I & 0 \\ [{}^{B}p_{o}] \cdot I & I \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} $$

Spatial Force (Wrench)

- Consider a rigid body with many forces on it and fix an arbitrary point O in space

- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O

    $$f = \sum_i f_i$$

  - A moment n_O about point O

    $$n_O = \underline{\sum_i (\overrightarrow{Op_i}) \times f_i}$$

recall:

$$V_q = V_O + \omega \times \overrightarrow{Oq}$$

- **Spatial Force (Wrench):** is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix} \quad \swarrow \text{wrench.}$$

· what if we change reference point from "O" to "q"

$$n_q = \sum_i (\overrightarrow{qp_i}) \times f_i = n_O + \sum_i \left(\overrightarrow{qp_i} \times f_i - \overrightarrow{Op_i} \times f_i \right)$$

by definition

$$= n_O + \sum_i (\overrightarrow{qp_i} - \overrightarrow{Op_i}) \times f_i$$

Spatial Force in Plücker Coordinate Systems

$= n_o + \sum_i \vec{qo} \times f_i$

$= n_o + \vec{qo} \times f$

$= n_o + f \times \vec{oq}$

- Given a frame {A}, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

convention: chose frame origin as reference point.

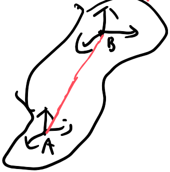
$$^A\mathcal{F} = \begin{bmatrix} ^A n_{O_A} \\ ^A f \end{bmatrix}$$

$$\boxed{n_q = n_o + f \times \vec{oq} = n_o + \vec{qo} \times f}$$

- Coordinate transform: $^A\mathcal{F} = {}^A X_B^* \, {}^B\mathcal{F}$ where $^A X_B^* = {}^B X_A{}^T$

Frame {A}, {B}, with $^AT_B = ({}^AR_B, {}^Ap_B)$

$^Af = \begin{bmatrix} ^A n_{O_A} \\ ^A f \end{bmatrix}$, $^Bf = \begin{bmatrix} ^B n_{O_B} \\ ^B f \end{bmatrix}$



- $^Af = {}^AR_B \, {}^Bf \cdots (1)$
- moment: coordinate-free: $\underbrace{n_{O_A} = n_{O_B} + (f \times (\vec{O_B O_A}))}$

^Bp_A

choose {A} frame to express: $^An_{O_A} = {}^AR_B \, {}^Bn_{O_B} + {}^AR_B \left({}^Bf \times {}^B(\vec{O_B O_A})\right)$

$= {}^AR_B \left({}^Bn_{O_B} + (-{}^Bp_A) \times {}^Bf\right) \cdots (2)$

$$\left.\begin{array}{l}①\\②\end{array}\right\} \Rightarrow \begin{bmatrix} {}^{A}n_{A} \\ {}^{A}f \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} {}^{A}R_{B} & -{}^{A}R_{B}[{}^{B}p_{A}] \\ \hline 0 & {}^{A}R_{B} \end{array}\right]}_{\triangleq\, {}^{A}X_{B}^{*}} \begin{bmatrix} {}^{B}n_{B} \\ {}^{B}f \end{bmatrix}$$

$$[a]^{T} = -[a]$$

$${}^{B}X_{A} = \left[\begin{array}{c|c} {}^{B}R_{A} & 0 \\ \hline [{}^{B}p_{A}]\,{}^{B}R_{A} & {}^{B}R_{A} \end{array}\right], \quad {}^{B}X_{A}^{T} = \left[\begin{array}{c|c} {}^{A}R_{B} & -{}^{A}R_{B}[{}^{B}p_{A}] \\ \hline 0 & {}^{A}R_{B} \end{array}\right]$$

$$\Rightarrow {}^{A}X_{B}^{*} = \left({}^{B}X_{A}\right)^{T}$$

Wrench-Twist Pair and Power

power = $f^T v = \langle f, v \rangle$

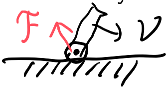
- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$
- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)
- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

$$\underbrace{P}_{\text{scalar}} = \underbrace{({}^A\mathcal{V})^T}_{1\times 6} \underbrace{{}^A\mathcal{F}}_{6\times 1} = {}^A\mathcal{F}^T\,{}^A\mathcal{V}$$

$$= \underbrace{({}^A\omega)^T ({}^A n_{o_A})}_{\text{rotational}} + \underbrace{{}^A v_{o_A}^T\,{}^A f}_{\text{power}}$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by: $\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$



- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \mathcal{V}^T \mathcal{F} = \boxed{(\hat{\mathcal{S}}^T \mathcal{F})}\dot{\theta} \triangleq \tau\dot{\theta}$$

$\hat{S}\dot{\theta}$

$= \triangleq \tau \triangleq \underset{6\times 1}{\hat{S}^T} \underset{6\times 1}{\mathcal{F}} = \mathcal{F}^T \hat{S}$

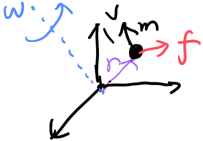
- $\tau = \hat{\mathcal{S}}^T \mathcal{F} = \mathcal{F}^T \hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Rotational Inertia (1/2)



- Recall momentum for point mass:

Linear motion: ⟷ rotational

velocity: $v = \dot{r}$, $a = \dot{v} = \ddot{r} \in \mathbb{R}^3$ ⟷ $\omega = \hat{\omega}\dot{\theta}$, $v = \omega \times r$

force: $f = ma = m\dot{v} = m\ddot{r}$ ⟷ moment: $n = r \times f$

Linear momentum: $L = m \cdot v$ ⟷ Angular momentum:

$$\phi = r \times L$$

$$= r \times (m\, \omega \times r)$$

$$= \underbrace{m}_{\text{scalar}}\, \underbrace{[r]}_{3\times3}\, \underbrace{[-r]}_{3\times3}\, \omega$$

$m[r][r]^T$

Inertia matrix

$$\phi = I\omega$$

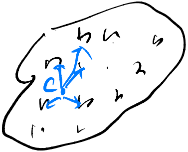
Rotational Inertia (2/2)

$\phi = \sum_i m_i [r_i][r_i]^T w$

- Rotational Inertia: $\bar{I} = \int_V \rho(r)[r][r]^T dr$ → point mass 做积分.
  - $\rho(\cdot)$ is the density function of the body

this matrix depends on coordinate system

  - $\bar{I}$ depends on coordinate system
  - It is a constant matrix if the origin coincides with CoM



What's def of Center of Mass:

$$c \triangleq CoM = \frac{1}{m}\int \rho(r)\, r\, dV$$

$$c \approx \frac{1}{m}\sum m_i r_i$$

If c is CoM, then $\frac{1}{m}\sum m_i (\vec{cr_i}) = 0$

$$\Rightarrow \sum m_i \vec{cr_i} = 0 \Rightarrow \sum m_i [\vec{cr_i}] = 0 \cdots (1)$$

Spatial Momentum

- Consider a rigid body with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C (derivation below works only when c = CoM)
  - Linear momentum: $L \triangleq m v_c$ velocity of CoM, why?

    $\underset{3\times 1}{}$

    $L = \sum_i m_i v_i$ point mass i's velocity

    $= \sum_i m_i (v_c + \omega \times \vec{cr_i})$

    $= \underbrace{\sum_i m_i v_c}_{m v_c} + \underbrace{\sum_i m_i \, cr_i \times (-\omega)}_{(\sum_i m_i [cr_i])\omega}$

  - Angular momentum about CoM:

    $\boxed{\phi_c \triangleq \bar{I}_c \omega} \Leftarrow \phi_c = \sum_i \vec{cr_i} \times (m_i v_i) = \sum_i \vec{cr_i} \times (m_i v_c + m_i \omega \times \vec{cr_i})$

    $= \sum_i (\vec{cr_i} \times m_i v_c) + \sum m_i \vec{cr_i} \times \omega \times \vec{cr_i} = \bar{I}_c \omega$

    $= \cancel{(\sum_i m_i [cr_i]) v_c} = 0$

  - Angular momentum about a point O:

    by definition

    $\phi_o = \sum_i \vec{or_i} \times (m_i v_i) \overset{\text{V.F.Y.}}{=} \phi_c + \vec{oc} \times L$

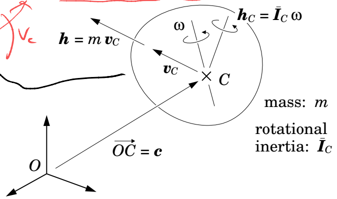
- Spatial Momentum:

  $h \triangleq \begin{bmatrix} \phi_{(o)} \\ L \end{bmatrix}$ refrence point

  convention: chosen to be origin of coordinate system.

  $\boxed{\phi_o = \phi_c + \vec{oc} \times L}$

  ② Coordinate free

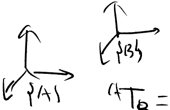


Change Reference Frame for Momentum

$\underline{V_q = V_c + \omega \times \vec{cq}}$

$n_q = n_c + \vec{qc} \times f$

- Spatial momentum transforms in the same way as spatial forces:



$$\underline{{}^A h = {}^A X_B^* \, {}^B h}$$

${}^A T_B = ({}^A R_B, {}^A p_B)$

$${}^B h = \begin{bmatrix} {}^B \phi_{O_B} \\ {}^B L \end{bmatrix}$$

$\boxed{{}^A \phi_{O_B} \text{ is a valid notion}}$

$${}^A L = {}^A R_B \, {}^B L$$

$${}^A h = \begin{bmatrix} {}^A \phi_{O_A} \\ {}^A L \end{bmatrix}$$

$$\underbrace{\phi_{O_A} = \phi_{O_B} + (\vec{O_A O_B} \times L)}_{\text{coordinate-free}} \quad (2)$$

$$\Rightarrow {}^A h = {}^A X_B^* \, {}^B h$$

$\Downarrow$ same as change of coordinate for wrench

Spatial Inertia

h : similar to force $\mathcal{F}$; think about inertia matrix as mapping from motion space M to Force space F

- Inertia of a rigid body defines linear relationship between velocity and momentum.

$\mathcal{V}$ twist

- Spacial inertia $\mathcal{I}$ is the one such that

$$h = \mathcal{I}\mathcal{V}$$

motion space : $\begin{cases} \text{twist } \mathcal{V} \\ \text{Accelleration } \mathcal{A} \end{cases}$

Force space : $\begin{cases} \text{wrench} \\ \text{momentum} \end{cases}$

- Let {C} be a frame whose origin coincide with CoM. Then

$$\underbrace{{}^C\mathcal{I}}_{6\times 6 \text{ matrix}} = \begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

I_3: 3x3 identity matrix

In this case, we know

${}^C\mathcal{V} = \begin{bmatrix} {}^C\omega \\ {}^Cv_c \end{bmatrix}$, ${}^Cv_{com} = {}^Cv_c \Rightarrow {}^CL = m\,{}^Cv_c$

${}^C\phi_c = {}^C\bar{I}_c\,{}^C\omega$ $\qquad {}^Ch = \begin{bmatrix} {}^C\bar{I}_c\,{}^C\omega \\ m\,{}^Cv_c \end{bmatrix} = \begin{bmatrix} {}^C\bar{I}_c & 0 \\ 0 & mI_{3\times 3} \end{bmatrix} \begin{bmatrix} \mathcal{V} \end{bmatrix}$

Spatial Inertia

${}^C R_B R_0 + {}^B R_c$

- Spatial inertia wrt another frame $\{A\}$:

$$\boxed{{}^A\mathcal{I} = {}^A X_C^* \, {}^C\mathcal{I} \, {}^C X_A}$$

$$ {}^A h = {}^A\mathcal{I} \, {}^A v = {}^A X_c^* \, {}^C h = {}^A X_c^* \, {}^C\mathcal{I} \, {}^C v = {}^A X_c^* \, {}^C\mathcal{I} \, {}^C X_A \, {}^A v $$

- Special case: ${}^A R_C = I_3$ (A's orientation is the same as $\{C\}$)

we know: ${}^A X_c = \begin{bmatrix} I_3 & 0 \\ [{}^A p_c] & I_3 \end{bmatrix}$

$$ {}^A\mathcal{I} = \begin{bmatrix} {}^C\mathcal{I} + m[{}^A p_c][{}^A p_c]^T & m[{}^A p_c] \\ m[{}^A p_c] & m I_{3\times 3} \end{bmatrix} $$

Outline

$v_3 = v_1 + v_2$

$^{B}\hat{f} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

X_B^*

$R_B = (x_1, y_1$

$X_B = (e_1, \cdots$

$$\hat{f} = [e^*_{B_1}\ \ e^*_{B_2}\ \cdots\ e^*_{B_6}]\ {}^{B}\hat{f}$$

basis vectors for wrench

Coordinate-free expression

- Spatial Acceleration

- Spatial Force (Wrench)

$[\hat{x}_A\ \ \hat{y}_A\ \ \hat{z}_A]$

$$\dot{\hat{f}} = \dot{X}_B^*\ {}^{B}\hat{f} + X_B^*\ ({}^{B}\hat{f})'$$

${}^{B}\dot{\hat{f}}$ de

- Spatial Momentum

~~$^{A}R_A = I$~~

It turns out (if {B} has velocity ν_B) then

$$\dot{X}_B^* = \nu_B \times^* X_B^*$$

- Newton-Euler Equation using Spatial Vectors

where "$\times^*$" defined as:

$$\nu = \begin{bmatrix} w \\ v \end{bmatrix}, \quad \hat{f} = \begin{bmatrix} n \\ f \end{bmatrix}, \quad \nu \times^* \hat{f} \triangleq \begin{bmatrix} w \times n + v \times f \\ w \times f \end{bmatrix} =$$

Cross Product for Spatial Force and Momentum

Assume frame A is moving with velocity ${}^{A}\mathcal{V}_A$

$\to \{A\}$ - twist

- $\underbrace{{}^{A}\left[\frac{d}{dt}\mathcal{F}\right]}_{\text{coordinate free}} = \underbrace{\frac{d}{dt}\left({}^{A}\mathcal{F}\right) + {}^{A}\mathcal{V}\times^{*}{}^{A}\mathcal{F}}_{\text{apparent derivative}}$

Recall: ${}^{A}\left(\frac{d}{dt}\mathcal{V}_{body}\right) = \frac{d}{dt}\left({}^{A}\mathcal{V}_{body}\right) + {}^{A}\mathcal{V}_A \times {}^{A}\mathcal{V}_{body}$

- ${}^{A}\left[\frac{d}{dt}h\right] = \frac{d}{dt}\left({}^{A}h\right) + {}^{A}\mathcal{V}\times^{*}{}^{A}h$

or equivalently

$$\underset{6\times 6}{[\mathcal{V}\times^{*}]} = \begin{bmatrix} [\omega] & [v] \\ 0 & [\omega] \end{bmatrix}$$

$$\dot{X}_B^{*} = [\mathcal{V}_B \times^{*}]\, X_B^{*}$$

you can choose "o" frame to derive this

Fact: $[\mathcal{V}\times^{*}] = -[\mathcal{V}\times]^{T}$

$$\underbrace{\dot{\mathcal{F}} = X_B^{*}\,{}^{B}\mathring{\mathcal{F}} + [\mathcal{V}_B \times^{*}]\, X_B^{*}\,{}^{B}\mathcal{F}}$$

$\Downarrow$

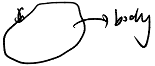
use $\{B\}$ ~~for~~ to express this physics

$${}^{B}\dot{\mathcal{F}} = {}^{B}\mathring{\mathcal{F}} + [{}^{B}\mathcal{V}_B \times^{*}]\,{}^{B}\mathcal{F}$$

$$\dot{\mathcal{V}}$$

Newton-Euler Equation

- Newton-Euler equation:



net wrench applied to the body

$$\mathcal{F} = \frac{d}{dt}h = \mathcal{I}\mathcal{A} + \mathcal{V} \times^* \mathcal{I}\mathcal{V}$$

coordinate-free : i.e. its form ~~dt~~ does not depend on coordinate-system

- Adopting spatial vectors, the Newton-Euler equation has the same form in any frame

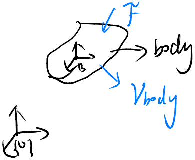
$$\mathcal{F} \doteq \frac{d}{dt}(h) = \frac{d}{dt}(\mathcal{I}\mathcal{V}) = \mathcal{I}\mathcal{A} + \dot{\mathcal{I}}\mathcal{V} \; ?$$

$$= \mathcal{I}\mathcal{A} + \mathcal{V} \times^* \mathcal{I}\mathcal{V}$$

due to $\mathcal{V}$ (velocity) is changing

due to inertia is changing

Let's work with inertia frame {o} to derive the NE-equation:



Assumptions:

- {B}: body fixed

$\Rightarrow \mathcal{V}_B = \mathcal{V}_{body}$, ${}^B\mathcal{I}$ constant.

Derivations of Newton-Euler Equation

- $${}^{o}\mathcal{F} = {}^{o}\left(\frac{d}{dt}h\right) = \frac{d}{dt}({}^{o}h) = \frac{d}{dt}({}^{o}\mathcal{I}\,{}^{o}\mathcal{V}) = {}^{o}\dot{\mathcal{I}}\,{}^{o}\mathcal{V} + {}^{o}\mathcal{I}\,{}^{o}\mathcal{A}$$

$$= \frac{d}{dt}\left({}^{o}X_B^{*}\,{}^{B}\mathcal{I}\,{}^{B}X_o\right){}^{o}\mathcal{V} + {}^{o}\mathcal{I}\,{}^{o}\mathcal{A}$$

$$= \underline{{}^{o}\dot{X}_B^{*}\,{}^{B}\mathcal{I}\,{}^{B}X_o\,{}^{o}\mathcal{V}} + \underline{{}^{o}X_B^{*}\,{}^{B}\mathcal{I}\,{}^{B}\dot{X}_o\,{}^{o}\mathcal{V}} + {}^{o}\mathcal{I}\,{}^{o}\mathcal{A}$$

$$= [{}^{o}\mathcal{V}_B\times^{*}]\left({}^{o}X_B^{*}\,{}^{B}\mathcal{I}\,{}^{B}X_o\right){}^{o}\mathcal{V}$$

$$\underline{- {}^{o}X_B^{*}\,{}^{B}\mathcal{I}\,{}^{B}X_o[{}^{o}\mathcal{V}_B\times]\,{}^{o}\mathcal{V}} + {}^{o}\mathcal{I}\,{}^{o}\dot{\mathcal{A}}$$

$${}^{o}\mathcal{V} = {}^{o}\mathcal{V}_B$$

$$= [{}^{o}\mathcal{V}_B\times^{*}]\,{}^{o}\mathcal{I}\,{}^{o}\mathcal{V} - 0 + {}^{o}\mathcal{I}\,{}^{o}\mathcal{A}$$

$$= {}^{o}\mathcal{I}\,{}^{o}\mathcal{A} + {}^{o}\mathcal{V}_B\times^{*}\,{}^{o}\mathcal{I}\,{}^{o}\mathcal{V}$$

$${}^{o}\mathcal{V}$$

choose any frame: $${}^{B}\mathcal{F} = {}^{B}\mathcal{I}\,{}^{B}\mathcal{A} + {}^{B}\mathcal{V}\times^{*}\,{}^{B}\mathcal{I}\,{}^{B}\mathcal{V}$$

side note

${}^{B}\dot{X}_o$?

- we know

$${}^{o}\dot{X}_B = [{}^{o}\mathcal{V}_B\times]\,{}^{o}X_B$$

$$({}^{o}X_B\,{}^{B}X_o)^{\cdot} = \mathcal{I}$$

$${}^{o}\dot{X}_B\,{}^{B}X_o + {}^{o}X_B\,{}^{B}\dot{X}_o = 0$$

$${}^{B}\dot{X}_o = -{}^{B}X_o\,{}^{o}\dot{X}_B\,{}^{B}X_o$$

$$= -{}^{B}X_o[{}^{o}\mathcal{V}_B\times]\cancel{{}^{o}X_B\,{}^{B}X_o}$$

$$= -{}^{B}X_o[{}^{o}\mathcal{V}_B\times]$$

More Discussions

- Review/summary:
  - Spatial accelleration: $\mathcal{A} \in \mathbb{R}^6$. $\begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix}$ coordinate-free.

$$\mathcal{A}_{body} \triangleq \lim_{\delta \to 0} \frac{\mathcal{V}_{body}(t+\delta) - \mathcal{V}_{body}(t)}{\delta}$$

$$\triangleq \dot{\mathcal{V}}_{body}$$

  - working with inertia/stationary frame:

$$\left({}^{o}\frac{d}{dt}\mathcal{V}_{body}\right) \Leftarrow {}^{o}\mathcal{A}_{body} = \underbrace{\frac{d}{dt}\left({}^{o}\mathcal{V}_{body}\right)}$$

  - working with moving frame: (suppose {B} is moving) ~~apparent derivative~~ ${}^{o}\dot{\mathcal{V}}_{body}$

$${}^{B}\mathcal{A}_{body} \neq \left({}^{B}\mathcal{V}_{body}\right)'$$

spatial cross product

$${}^{B}\mathcal{A}_{body} = \frac{d}{dt}\left({}^{B}\mathcal{V}_{body}\right) + {}^{B}\mathcal{V}_{B} \times {}^{B}\mathcal{V}_{body}$$

$${}^{B}\mathcal{A}_{body} = {}^{B}X_{o}\left({}^{o}\mathcal{A}_{body}\right)$$

$$\mathcal{V}\times :\triangleq \underbrace{[\mathcal{V}\times]}_{6\times 6} = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}$$

$$\underbrace{[\mathcal{V}]}_{4\times 4} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

$\{A\}$

- Recall: $\dot{R}_A = \omega_A \times R_A$, $[R\omega] = R[\omega]R^T$

corresponding: ${}^0\dot{X}_A = \underbrace{{}^0\mathcal{V}_A \times}\, {}^0X_A$, $[X\mathcal{V}\times] = X[\mathcal{V}\times]X^T$

6x1

More Discussions

coordinate free.

- Review / Summary.

-spatial acceleration : $A \in \mathbb{R}^6$. $A_{body} \triangleq \lim_{\delta \to 0} \frac{V_{body}(t+\delta) - V_{body}(t)}{\delta} \triangleq \dot{V}_{body}$.

- working with inertia/stationary frame:

$${}^0\left(\frac{d}{dt} V_{body}\right) \Leftarrow {}^0A_{body} = \frac{d}{dt}\left({}^0V_{body}\right).$$

apparent derivate ${}^0\dot{V}_{body}$

- working with moving frame: (suppose {B} is moving).

$${}^BA_{body} \neq \left({}^BV_{body}\right)'$$

spatial cross product.

$${}^BA_{body} = \frac{d}{dt}\left({}^BV_{body}\right) + \boxed{{}^BV_B \times} {}^BV_{body}$$

$${}^BA_{body} = {}^BX_0 \left({}^0A_{body}\right)$$

screw axis

$$\underset{4\times 4}{[V]} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

$$V\times \triangleq \underset{6\times 6}{[V\times]} = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix}$$

More Discussions

- Recall: $\dot{R}_A = w_A \times R_A$. $[Rw] = R[w]R^T$

  coordinate free.

  corresponding: ${}^0\dot{X}_A = \underline{{}^0V_A \times}\, {}^0X_A$. $[XV\times] = X[V\times]X^T$.

  spatial cross product.

- spatial force/wrench: ${}^B\mathcal{F} = \begin{bmatrix} {}^Bn_{O_B} \\ {}^Bf \end{bmatrix}$ ${}^A\mathcal{F} = {}^AX_B^* \, {}^B\mathcal{F}$

$${}^0\dot{X}_A^* = V_A \times^* \, {}^0X_A^*$$

$$= \underbrace{[V_A \times^*]}_{6\times 6} \, {}^0X_A^*$$

$${}^AX_B^* = ({}^BX_A)^T$$

- Joint torque.

$$\tau\dot{\theta} = \mathcal{V}^T\mathcal{F} = (S^T\dot{\theta})\,\mathcal{F}$$

$$\tau = S^T\mathcal{F} = \mathcal{F}^T S$$

More Discussions

spatial momentum:

$$ {}^{A}h = \begin{bmatrix} {}^{A}\phi_{o_A} \\ {}^{A}L \end{bmatrix}, \quad {}^{A}h = {}^{A}X_{B}^{*}\, {}^{B}h $$

- spatial inertia matrix @ (CoM frame)

$$ {}^{C}\mathcal{I} = \begin{bmatrix} {}^{C}\bar{I} & 0 \\ 0 & mI_3 \end{bmatrix} $$

$$ {}^{A}\mathcal{I} = {}^{A}X_{C}^{*}\, {}^{C}\mathcal{I}\, {}^{C}X_{A} $$

NE:

$$ \mathcal{F} = \frac{d}{dt}(h) = \mathcal{I}A + \mathcal{V}\times^{*}\mathcal{I}\mathcal{V} $$