MEE5114 Advanced Control for Robotics

# Lecture 11: Differential Inverse Kinematics

**Prof. Wei Zhang**

**CLEAR Lab**
Department of Mechanical and Energy Engineering
Southern University of Science and Technology, Shenzhen, China
https://www.wzhanglab.site/

# Inverse Kinematics Problem

PoE

end-effecter pose

- Forward kinematics: $\theta \to T(\theta) = (\underset{\sim}{R(\theta)}, p(\underset{\sim}{\theta})) = e^{[{}^{0}S_1]\theta_1} e^{[{}^{0}S_1]\theta_1} \cdots e^{[{}^{0}S_n]\theta_n} M$

  Joint variable $\theta_1, \cdots, \theta_2$

- Often times, we are interested in certain aspect of the pose:
  represented by task space variable. $x$ (e.g. $x$ is RPY, or origin of $T(\theta)$)
  $x = f(T(\theta))$
- Inverse Kinematics: Given desired $x_d$, find $\theta_d$ such that $x_d = f(T(\theta_d))$ → solve for $\theta_d$ for given $x_d$.

  solve the nonlinear equation

- Analytical solution vs numerical solution:
  - In some special case, $\theta_d$ can be found analytically.
  - For general problems, relie on numerical solvers.
    IkFast.

# Differential IK (1/3)

FK: map joint position $\theta \rightarrow$ e.f. pose $T(\theta)$

DK: . . . . . . . . velocity $\dot{\theta}$ . . . . . velocity $\mathcal{V}$

- Differential Kinematics:
  - Geometric Jacobian:

Geometric Jacobian

$$\mathcal{V}_e = J(\theta)\dot{\theta}$$

end-effector twist

  - Analytical Jacobian:

For task space variable $x$, $x = f(T(\theta))$, $\dot{x} = \frac{\partial(f(T(\cdot)))}{\partial\theta}\dot{\theta}$

$$\dot{x} = J_a(\theta)\dot{\theta} \quad ; \quad J_a(\theta) = E(\theta)J(\theta)$$

- Differential IK: Given desired task space velocity $\dot{x}_d$, find $\dot{\theta}_d$ such that

$$\dot{x}_d = J_a(\theta)\dot{\theta}_d$$

decision variable

$\in \mathbb{R}^{n_x \times n}$

joint number

task space variable dimension

~~Let~~ $\Delta x = 6$

# Differential IK (2/3)

- **Singularity**:

- ③. $J_a(\theta) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$n_x = 2, \; n = 3$

$\mathrm{rank}(J_a(\theta)) = 2$

$\mathrm{col}(J_a(\theta)) = \mathbb{R}^2$

- ④ $J_a(\theta) = E(\theta) J(\theta)$

if $J(\theta)$ is rank defficient ~~singular~~ kinematically singular

if $J(\theta)$ is full rank, $E(\theta)$ is rank defficient : reprentation singularity

- ① $J_a(\theta)\dot{\theta}$ : viewed as linear combination of columns of $J_a(\theta)$

we can choose $\dot{\theta}$ to produce different $\dot{x} = J_a(\theta)\dot{\theta}$

Feasible/possible task velocity. $\dot{x}$ lies in the column space of $J_a(\theta)$

- ② Singularity : $J_a(\theta)$ is not full row rank.

e.g. $J_a(\theta) = \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow$

$n = 6, \; n_x = 2$

$\mathrm{col}(J_a(\theta)) = \mathrm{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, can only produce velocity of the form $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, $\alpha \in \mathbb{R}$

# Differential IK (3/3)

- Solution of Differential IK: $\dot{x}_d = J_a(\theta)\dot{\theta}$ → variable to be found.

- If $\dot{x}_d \in$ column space $col(J_a(\theta))$, then there is a solution.

- $\dot{x}_d \notin$ column space $col(J_a(\theta))$, then we typical find "best effort" solution

- $\dot{\theta}_d = J_a^+(\theta)\dot{x}_d$ → pseudo-inverse solution

① : full rank (typically $n_x < n$)

$$J_a^+(\theta) = \left(J_a^T (J_a J_a^T)^{-1}\right) \leftarrow \text{right inverse}$$

$$J_a(\theta) J_a^+(\theta) = I_{n_x \times n_x}$$

- In general, $J_a^+(\theta)\dot{x}_d$ is the solution to

$$\boxed{\min_{\dot{\theta}} \|\dot{x}_d - J_a(\theta)\dot{\theta}\|^2}$$

# Optimization-Based Differential IK

- Given task space velocity $\dot{x}_d$, find $\dot{\theta}$

# Differential IK with Constraints

- Given task space velocity $\dot{x}_d$, find $\dot{\theta}$ with constraints $\theta_i \in [\theta_i^-, \theta_i^+]$, and $\dot{\theta}_i \in [\dot{\theta}_i^-, \dot{\theta}_i^+]$, $i = 1, \ldots, n$

– Find $\dot{\theta}$ such that $\underbrace{\theta_i^- \le \theta_i \le \theta_i^+ , \ \dot{\theta}_i^- \le \dot{\theta}_i \le \dot{\theta}_i^+ , \ \dot{x}_d \approx J_a(\theta)\dot{\theta}}$

Given $\theta$, (current position) $\Rightarrow J_a(\theta)$

– solve this via optimization:

$$\min_{\dot{\theta} = (\dot{\theta}_1, \ldots, \dot{\theta}_n)} \| \dot{x}_d - J_a(\theta)\dot{\theta} \|^2$$

$$\begin{cases} \dot{\theta}_i^- \le \dot{\theta}_i \le \dot{\theta}_i^+ , \quad i = \cdots n \quad \text{(1a)} \\ \theta_i^- \le \theta_i + \dot{\theta}_i \cdot \Delta t \le \theta_i^+ \quad \text{(1b)} \end{cases}$$

optimization variable

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

# More Discussions

- General form:

$$\min_{\dot\theta} \|\dot{x}_d - J_a(\theta)\dot\theta\|^2 + \underbrace{\text{regularization term}}_{\text{e.g. } \rho\|\theta+\dot\theta\Delta t - \theta_d\|^2}$$

subj: (1a)

(1b)

- DIK:

$$\dot{x}_d \longrightarrow \boxed{\text{DIK}} \longrightarrow \dot\theta_d$$

· How to find $\dot{x}_d$?: outer-loop control (planning)

- Example 1: Given $T_b = (R_b, p_b)$ current pose of end-effector

· Let $x = \begin{bmatrix} p_{b,x} \\ p_{b,y} \\ p_{b,z} \end{bmatrix} \in \mathbb{R}^3$, current position of $\{b\}$

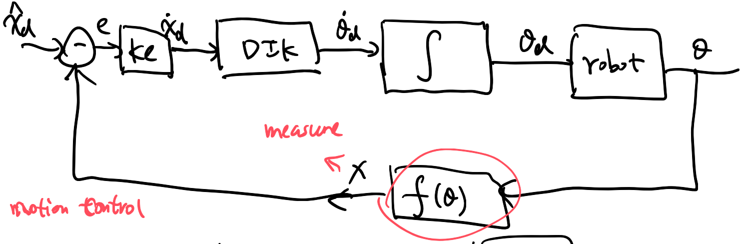
# More Discussions

- · we are given $\hat{x}_d = \begin{bmatrix} \hat{p}_{d,x} \\ \hat{p}_{d,y} \\ \hat{p}_{d,z} \end{bmatrix}$, desired location of origin $\{b\}$

  · Typical approach. use IK to find $\hat{\theta}_d = IK(\hat{x}_d)$

  $\Rightarrow$ apply $\hat{\theta}_d$

· Using DIK.




$e \doteq \hat{x}_d - x$

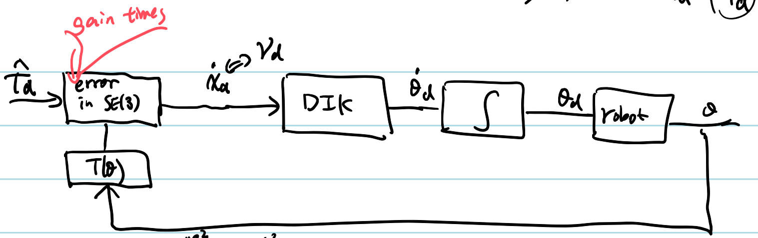
$\boxed{\dot{x}_d = k \cdot e}$

- outer Loop motion Control

· We will show in Next lec. by choosing proper $k$ $\boxed{\dot{x}_d = ke}$ can make $x \to \hat{x}_d$

- In this case, DIK serves as a simple motion controller.

- Example 2: current pose $T_b(\theta) = (R_b(\theta), p_b(\theta))$, desire $\hat{X}_d = \hat{T}_d \in SE(3)$




- Euclidean space: $\underset{\in\mathbb{R}^3}{e} = \underset{\in\mathbb{R}^3}{\hat{x}_d} - \underline{x \in \mathbb{R}^3}$

- SE(3): $T_{error} \neq \hat{T}_d - T$

$$e^{[S]\beta} \cdot {}^0T_b(\theta) = {}^0\hat{T}_d$$

$$\Rightarrow \underset{SE(3)}{e^{[S]\beta}} = {}^0\hat{T}_d \, {}^0T_b^{-1}(\theta)$$

$$\underset{se(3)}{[S]\beta} = \log({}^0\hat{T}_d \, {}^0T_b^{-1}(\theta))$$

$$\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \rightarrow \underline{\begin{bmatrix} \omega \\ v \end{bmatrix} = S}$$

$$\Downarrow$$

$$\Rightarrow \mathcal{V}_d = k \cdot S$$